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ANALYTICAL MECHANICS OF SYSTEMS WITH VARIABLE MASS

V. S. Novoselov

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FOREWORD:

The solution of problems of rocket dynamics and celestial ballistics, as well as problems of industry and transportation may call for analytical methods in mechanics with variable mass. The author of this book acts himself the task of acquainting the reader with the basic analytical methods for mechanical systems with variable mass and demonstration of the application of these methods to concrete problems.

Prt of this book includes lectures for a special course, read during the ten years in the mathematical - mechanical faculty at the Leningrad University. The book also includes special questions, reduced to direct application. This book will help the students of higher courses and post-graduate students in universities and technical colleges/institutes to master the basic principles and analytical characteristics in this new department of mechanics. Consideration of special questions, namely: motion of a point in the central field, motion of gyroscopic systems and vehicles with variable mass, make this book very useful for those employed in Scientific Research Institutes and construction departments.

It should also be emphasized, that the questions under study are more of theoretical nature as opposed to practical. In essence, at present, the mechanics of bodies of variable mass is of great practical value in the studies of rocket ballistics. There

exists an established theory[3] which can fully meet the requirements, not only of the evolving of equations for the jet propulsion, but also for their substantiation. This book has been written using the results of analysis carried out by the author but it does not claim to serve as an exhaustive monography for all the analytical methods in the mechanical systems of variable mass. The author will be very grateful to readers, who can send their critical remarks care of the Publishers. The author is specially gr teful to S.S. Lavrov, G.N. Duboshin, M.Z. Kolovsky and S.N. Kirpichnikov for their kind criticism of the manuscript of this book.

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INTRODUCTION:

This book is dedicated to the theory of the analytical system of mechanics that use jet propulsion. Such propulsions are accompanied by separation, addition or simultaneous separation and addition of particles.

Let us, very briefly, dwell upon the basic historical stages of development of mechanical systems having variable mass.

In 1750, a Hungarian scientist, Yanosh Segner invented a wheel that brought into motion, by recoil force, the water jet. Segner's wheel turned out to be one of the first applications of the principle of jet propulsion. The other fairly ancient application of this principle could be the rockets for fireworks and signalling. Russian inventor K.I. Konstantinov, developed construction plans for combat missiles in the middle of the XIX century. The first practical application of the jet propulsion principle belongs to the last quarter of the XIX century, when ship construction in Sweden and England used pumps to reduce the on-coming water from the bow, to a certain pressure and then propelled this water out through a nozzle fitted in the stern.

The study of the motion with variable mass is also the subject of Cayley's problem (1858). The motion of a chain and the problem of moving two bodies having variable mass were studies by Dufour (1866), Opplotzer (1884) and Hulden (1884).

The work of I.V. Meschersk̆ii[53,54], during the period 1897 to 1904, has laid the basis for the development of mechanical systems with variable mass as a special section of theoretical mechanics. Meschersk̆ii introduced the conception of a point with variable mass, derived equations for the motion of a point of variable mass and carried out experiments to prove this equation for a series of problems, taken as the material point of variable mass, assuming that the added and separated particles did not shift within the body.

The works of K.E. Tsiolkovskii (specially of 1903) on questions of astronautics[104] were of great importance in the establishment of a system of mechanics having variable mass. In the thirties of the XX century a whole series of research on different problems of variable mass were published: G.N. Duboshin[15 - 19] (1926 - 1930): Levi-Civita (1928); N.A. Rindin[86] (1929 - 1932); V.V. Stepanov[100] (1930); V.P. Vetchinkin[8] (1935); V.S. Zuev[24] (1936); M.K. Tikhonarov[102] (1936).

The second World War and specially the Patriotic War of the Soviet people against Hitleristic aggressors, where for the first time, rocket weapons were widely used, gave a fresh stimulus of the development of mechanical systems with variable mass.

The results of military experiments of foreign authors R. Renkin; I. Cooy and I. Youthenbogart; R. Resser; R. Newton and G. Gross were published in the Russian language in monographs[32,82,83], and relate to 1950-1951.

In post-war years, fundamental work by the Soviet Scientists have been published, namely: D.E. Okhotzimskii[76] (1946) A.U. Ishlinskii[25] (1946), A.A. Kosmodemyanskii[33 - 36] (1946 - 1947), F.R. Gantmakher and L.M. Levin[11,12] (1947), Ya. G. Panovko and N.V. Butenin[78] (1947), A. I. Lurier[45] (1948), I.P. Ginzburg[13] (1949), these works are the results of strenuous research by these scientists during the Patriotic War (Second World War).

The works of Okhotzimskii and Ishlinskii play an important role in the development of optimum trajectory in the central field. In the works of Kosmodemyanskii, Lurier, Panovko and Butenin systematic description of mechanics of a solid body of variable mass are given. The works of Gantmakher and Levin are dedicated to the study of equations of propulsion of actual rockets, that have an inner movement of particles. The specially important contribution by Gantmakher in the development of mechanical systems with variable mass, must be mentioned. Having carried out a deep study of the influence of inner movement of particles, Gantmakher[11,12] comes to the integral formulae for the principal vector and principal moment of reactive forces.

During the last ten-fifteen years, numerous studies have been carried out, for both, separate problems in the motion of systems with variable mass as well as for problems of developing an analytical mechanics for these systems. Since, in the Soviet Union, these analysis were carried out more intensely, it is only possible to mention the names of some of the Soviet Authors: N.A. Slezkin[98, 99]

(1951), V.F. Kotov[37] (1955), V.S. Novoselov[57 - 59, 62-75] (1956-1965) V.A. Shapa[56, 88-97] (1956-1961), M. Sh. Amionov[1,2] (1958, 1959), V.M. Karagodin[29,30] (1955-1963), A.I. Lurie[46,47] (1961, 1964), N.G. Magnaradze[49-51] (1959-1961), F.A. Appazov, L.S. Lavrov, V.P. Mishin[3] (1966). Importance was given to the studies of the seminerien A.I. Artobolevskii at the institute of machinery regarding problems of mechanical motion, taking into consideration the variable mass of elements and products under fabrication.

The development of contemporary technology and its qualitative changes have brought about wide applications of jet propulsion that require solution of new problems in propulsion with variable mass. When setting and analysing such problems, it is worth while to note, the relatively small contributions that where achieved through the development of analytical methods of study of propulstion systems with variable mass and the solution of certain concrete problems.

This book, which consists of eight chapters, deals with the basic questions of analytical mechanics of systems with variable mass. Chapter I is dedicated to the study of the motion of a point having variable mass and at the same time taking into consideration the effect of the inner motion of particles. Chapter II discusses the laws of motion in a mechanical system and solid bodies with variable mass. Chapter III and IV clarify the analytical properties of equations of motion in holonomic and non-holonomic systems with variable mass. Chapter V contains a description of special characteristics of mechanical bodies with variable mass. Here formulae are evolved for the principal vector and principal moment of reactive forces, as well as for the angular rotation velocity of the main inertial axis in the

body of variable Mass. Chapter VI studies the analytical properties of differential equations, describing the motion of gyroscopic bodies with variable mass. Chapter VII is dedicated to the problem of selecting the optimum trajectory of motion of a body with variable mass in the central field. Here, only those questions are discussed which are directly connected with the description of motion of a body with variable mass, by setting various problems. Chapter VIII deals with the study of stability of one-wheeled vehicles of variable mass and also discusses the movement of sledges and automobiles of variable mass.

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CHAPTER-I

MOTION OF A POINT HAVING VARIABLE MASS.

1. Equation of Motion of the Point Having Variable Mass.

1. General Remarks:- It is necessary to observe that variation of mass is not to be understood as its creation or disappearance, it rather characterizes the process of addition to or ejection from the body of particles. Our object of study would be a system of particles with constant masses. Certain particles leave the system under study and new particles take their place.

Mechanics is concerned with obtaining the equations of motion of the changing system of particles and their study.

It is known that the theoretical mechanics in conjunction with the existing ideas about matter (material) point has accomplished a considerable analytical simplification of the structure. It is possible to develop a complete course of the mechanics of system with constant masses based on the Newton's differential equations of motion of matter point through logical discussions and analytical computations.

This method can be used with advantage for the systems having variable masses. However, the application of this method requires the concepts of matter point having variable mass capable

of showing the complex chatacter of inter-action of the body consisting of th moving particles. For this it is convenient to adopt the usual interpretation of kinematics of a point and of a solid body and make use of the generalized coordinates and the ideal constraints.

This approach enjoins a study of the point having variable mass as a set of particles which, at a time t, are found in a region limited by a certain controlled surface. This region, according to the problem, may be supposed to be moving together with some geometrical point. The motion of the specified particles may be entirely different from the motion of the (associated) geometrical point. It will be useful to link with the moving region the body which will work as the frame, and the motion of the particles will be examined accordingly. If the solid body itself whose size or rotatory motion are neglected is considered in place of the point having variable mass, the outer covering of the body will serve as the frame.

In complicated dynamical problems where the translatory as well as the rotatory motions of the solid bodies having variable mass are studied, each body must be mentally broken up into infinite number of small regions whose velocities coincide with the velocities of the the corresponding geometrical points belonging to the coordinate system, firmly connected with the frame of the body. These geometrical points (i.e., the moving regions with changing composition of the particles) will be identified as points having variable mass comprising the body under study. The particles will

pass through these "points" causing a change in their masses. However, the velocities and the accelerations of these "points", in each case, will conform to the laws of the solid kinematics.

The suggested approach in the frame-work of the structure of the mechanics of systems with constant masses sets out the fundamental laws and makes it possible to investigate the analytical features of the motion of the system.

2. Derivation of the equations of motion:- We will develop the equations of motion of a point having a variable mass [57]. Consider a system of the particles with constant mass. The system consists of particles which, at a time t, build the matter point and also of those particles which join the matter point in the interval Δt. To this system apply the rule of change of the principal vector of momentum.

At time t, the principal vector of momentum of the system is equal to

$$\bar{Q}(t) = \sum_i \Delta\mu_i \bar{u}_i^1 + \sum_j m_j \bar{u}_j(t) ,$$

where the first summation ranges over such particles which in time Δt, join the matter point (masses of these particles are denoted by $\Delta\mu_i$) and the second summation is taken over those particles which, at instant t, make the matter point. Their masses are denoted by m_j and their velocities by $\bar{u}_j$. $\bar{u}_j^1$ is the velocity of the jth particle which, at instant t, joins the matter point.

The principle vector of the moment of the system at instant

$t + \Delta t$ is

$$\bar{\cdot}(t + \Delta t) = \cdot_i \cdot \Delta \cdot_i \bar{u}_i^2 + \cdot_j n_j \bar{u}_j (t + \Delta t).$$

where $\bar{u}_i^2$ is the velocity of the ith particle at instant $t + \Delta t$ after joining the matter point.

We assume that the velocities of some particles making the matter point at instant t and entering in the second summation are not only changing continuously suddenly. Therefore, we will assume that the general mass of particles, $\cdot_j$, will be infinitely small, i.e., it tends to zero as Δt tends to zero.

Consider the motion of an auxiliary system of coordinates in such a way that some points of the system coincide with the matter point throughout the motion. Then the vectorial value of the continuous change of velocity of j-th particle in time t is equal to

$$(\cdot_e + \cdot_r^j + \cdot_c^j)\,\Delta t + \varepsilon_2,$$

where $\cdot_e$ is the acceleration of the matter point, $\cdot_r^j$ and $\cdot_c^j$ are relative (average) and coriolisc accelerations of i-th particle in respect of the auxiliary system. ε_2 is infinitesimal of the second order. Coriolisc acceleration can be expressed as:

$$\cdot_c^j = 2(\cdot \times \bar{u}_r^j),$$

in which $\cdot$ is the angular velocity of rotation of the auxiliary system of the coordinates, and $\bar{u}_r^j$ is average (relative) velocity of j-th particle.

Since the velocities of the particles comprising the matter point at instant t may change impulsively, therefore;

$$\sum_j m_j \Delta \bar{u}_j = \sum_j m_j (\bar{\omega}_e + \overset{-j}{\omega}_r + \overset{-j}{\omega}_C) \Delta t -$$

$$- \sum_k m_k (\bar{\omega}_e + \overset{-k}{\omega}_r + \overset{-k}{\omega}_C) \Delta t + \sum_k m_k (\bar{u}_k^2 - \bar{u}_k^1) + \bar{o}_2 \quad \ldots\ldots(1.1)$$

whose velocities undergo impulsive changes in time Δt. $\overset{-1}{u}_k$ and $\overset{-2}{u}_k$ are velocities of k-th particle at instants t and $t + \Delta t$ respectively.

Put $m_k = \Delta\mu_k$

Then the increment in the principal vector of moment in time Δt will be equal to :

$$\Delta \bar{q} = \bar{q}(t+\Delta t) - \bar{q}(t) = \sum_i \Delta\mu_i (\overset{-2}{u_i} - \overset{-1}{u_i}) + \sum_j m_j (\bar{\omega}_e + \overset{-j}{\omega}_r + \overset{-j}{\omega}_C) \Delta t -$$

$$- \sum_\kappa \Delta\mu_k (\bar{\omega}_e + \overset{-k}{\omega}_r + \overset{-k}{\omega}_C) \Delta t + \sum_\kappa \Delta\mu_k (\bar{u}_k^2 - \bar{u}_k^1) + \bar{o}_2 .$$

where $\overset{-k}{\omega}_r$ and $\overset{-k}{\omega}_C$ are relative (average) and the Coriliso accelerations of these particles till or after the required impulsive change in the velocity occurs. We have:

$$\frac{\Delta \bar{q}}{\Delta t} = \bar{\omega}_e \sum_j m_j + \sum_j m_j \overset{-j}{\omega}_C + \sum_j m_j \overset{-j}{\omega}_r +$$

$$+ \sum_i \frac{\Delta\mu_i}{\Delta t} (\bar{u}_i^2 - \bar{u}_i^1) + \sum_\kappa \frac{\Delta\mu_k}{\Delta t} (\bar{u}_k^2 - \bar{u}_k^1) + \bar{o}_1 .$$

we note that under the assumption

$$\sum_k \Delta\mu_k \to 0 \text{ as } \Delta t \to 0.$$

and 0_1 denotes the infinitesimal of the first order.

Since, by virtue of the law of change of the principal vector of momentum

$$\frac{\Delta \bar{Q}}{\Delta t} \to \bar{F}, \quad \text{when } \Delta t \to 0$$

where $\bar{F}$ denotes the principal vector of the external forces, acting on the matter point, then assuming the existence of the derivatives, we get

$$m \frac{d\bar{v}}{dt} = \bar{F} + \bar{R} \qquad \ldots\ldots(1.2)$$

Here $m = \sum_j m_j$ is the mass of the matter point and $\frac{d\bar{v}}{dt} = \bar{w}_e$ is its acceleration. $\bar{R}$ will be called the reactive force.

Reactive force R is equal to

$$\bar{R} = \bar{J} - \sum m\bar{w}_r + \bar{\Phi} \qquad \ldots\ldots(1.3)$$

In (1.3), $\quad \bar{J} = -2\sum \bar{\omega} \times m\bar{u}_r, \qquad \ldots\ldots(1.4)$

Summation in (1.3) and (1.4) are taken over all the particles forming the matter point at instant t.

$\bar{J}$ will be referred to as Coriliso force.

will be called impulsive force and is given by

$$\Phi = -\sum \frac{d\mu}{dt} \left(\bar{u}^2 - \bar{u}^1 \right) \qquad (1.5)$$

where the summation is carried over all the directions around the matter point, in which the impulsive change of the velocity takes place; $\frac{d\mu}{dt}$ is the intensity of the flow of particles in a given direction.

Putting

$$\bar{u}^2 - \bar{u}^1 = \Delta \bar{u}$$

we obtain the impulsive force in the form

$$\bar{\Phi} = -\sum \frac{d\mu}{dt} \Delta \bar{u} \qquad (1.6)$$

$\because \Delta \bar{u} = \Delta \bar{u}_r$, where $\Delta \bar{u}_r$ is the impulsive change of relative (average) velocity of particle at a given instant t,

$$\therefore \bar{\Phi} = -\sum \frac{d\mu}{dt} \Delta \bar{u}_r \qquad (1.7)$$

the impulsive force may be defined as the surface integral as

$$\bar{\Phi} = \iint_{(\sigma)} \bar{\varphi} \, d\sigma$$

over full solid angle $\sigma = 4\pi$ around the matter point, function $\bar{\varphi}$ signifies the specific value

$$\frac{d\mu}{dt} \Delta \bar{u}_r .$$

In view of the derivation of equation (1.2) we may give another definition of a point having variable mass. It is such a moving system of the particles with constant mass which is concentrated

during the motion in a region moving with a certain geometrical point belonging to the auxiliary system of the coordinates. The size of the moving system (body) can be neglected.

If, a solid body having variable mass whose size is neglected is considered in place of a point, the coordinate system connected with the body can be chosen as the auxiliary system. In this case, the average velocities and accelerations related to the body will be equal to the average velocities and the accelerations in respect of the auxiliary system and the Coriliso accelerations will also be the same.

Any system satisfying the requirements of our definition can be chosen as the auxiliary system. There are two possible auxiliary systems referred to the system I and system II. They must have one general point with which the matter point coincides. Since we neglect the geometrical dimensions of the region, investigated by means of material point having a variable mass, we must also neglect the rotatory velocities of the geometrical points belonging to the region, compared with the velocity of the translatory motion, which is equal to the velocity $\bar{u}$ of the material point. Therefore, we write

$$\bar{u}_{r1} = \bar{u}_{r2} = \bar{u}_r,$$

$$\bar{w}_{e1} = \bar{w}_{e2} = \frac{d\bar{v}}{dt}.$$

Here, as before, the relative characteristics are determined in the auxiliary coordinate system.

The Coriliso accelerations can be expressed as

$$\bar{\omega}_{C1} = 2\,(\bar{\omega}_1 \times \bar{u}_r\,), \quad \bar{\omega}_{C2} = 2\,(\bar{\omega}_2 \times \bar{u}_r\,),$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are the angular velocities of the rotation of system I and II.

When the system II is obtained conditionally by the addition of accelerations, we have

$$\bar{\omega}_{r2} = \bar{\omega}_{r1} + 2\,(\bar{\omega}_1 - \bar{\omega}_2\,) \times \bar{u}_r + \bar{\omega}^{e}_{12}.$$

in which $\bar{\omega}^{e}_{12}$ denotes the acceleration of that point of system I, with which the motivating particle coincides at instant t. Since the acceleration $\bar{\omega}^{e}_{12}$ in system II must be calculated, and all studied particles are found clustered in the general point determined by systems I and II, it follows that $\bar{\omega}^{e}_{12} = 0$. We get

$$\bar{\omega}_{r2} = \bar{\omega}_{r1} + 2\,(\bar{\omega}_1 - \bar{\omega}_2\,) \times \bar{u}_r.$$

therefore,

$$\bar{\omega}_{r1} - \bar{\omega}_{C1} = \bar{\omega}_{r2} + \bar{\omega}_{C2}. \qquad (1.8)$$

Impulsive force does not depend upon the selection of the coordinate system, we may, therefore, conclude from (1.8) that the impulsive forces do not change for different auxiliary systems.

REMARKS: Formula (1.3) expressing the reactive force has inner contradictions. It lies in the fact that very often the process of change of mass m may as well be regarded as the process of impulsive and continuous change of velocity with a considerable acceleration. It can be seen from the derivation of equation(1.3) for impulsive process which is taken into account in determining Φ. By the quantity $\vec{w}_r$ in (1.3) we have the average acceleration when the velocity continuously changes, i.e., if there be no impulsive change of velocity the particle would have the above specified acceleration. Since the mass of particles Δt is infinitesimal of the order of Δt, therefore, any finite change in the specified average acceleration does not lead to the change of the reactive force.

From here it follows that when the process of change of mass is regarded as the process of impulsive and continuous change of velocity, it considerably effects either the impulsive force (in this case the corresponding relative acceleration may be supposed to be zero) or the component $\sum m\vec{w}_r$, where $\vec{w}_r$ denotes the total relative acceleration. In order to avoid the difficulties in determining the reactive force, the strong effects are not taken into account. It is expedient to treat any process of change of mass, for which the average acceleration of the total change of velocity, however, large has relevance, as a non-impulsive process.

3. EXAMPLES: We will discuss two examples illustrating the application of the equation of motion of a point having a variable mass.

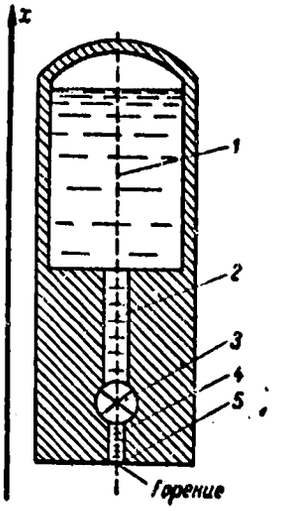

Рис. 1

Fig.1

1). We will investigate the ascending motion of a rocket whose construction is shown in Fig.1. The large vessel 1 contains liquid fuel (combustible liquid) which is sucked by the pump 3 through the canal 2 with an average velocity of $\bar{v}_2^r$ and an average acceleration of $\bar{w}_2^r$. Then the liquid pushes out through a narrower canal 4 with an average velocity of $\bar{u}_4^r$, and average acceleration of $\bar{w}_4^r$. In the nozzle 5 the fuel gets heated and leaves the rocket with an average velocity of $\bar{w}_5^r$.

It is convenient to take the system connected with the body (frame) of the rocket as the auxiliary system. It is supposed that the dimensions of the nozzle are negligibly small.

The reactive force will be equal to $\bar{R} = -m_2\bar{w}_2^r - m_4\bar{w}_4^r + \bar{\Phi}$, where m_2 - mass of the liquid present in canal 2 m_4 - mass of liquid present in canal 4, we neglect the average velocities and average

acceleration of the particles in the vessel I. The Coriliso accelerations will be equal to zeor as a consequence of the supposed ascending motion of the rocket. We have

$$\Phi = -\frac{d\mu}{dt}\bar{v}_2^r - \frac{d\mu}{dt}(\bar{v}_4^r - \bar{v}_2^r) - \frac{d\mu}{dt}(\bar{v}_5^r - \bar{v}_4^r) = \frac{d\mu}{dt}\bar{v}_5^r,$$

where $\frac{d\mu}{dt}$ denotes rate of fuel expenditure. We may write

$$\frac{d\mu}{dt} = \frac{dm}{dt}.$$

In the given case,

$$\Phi = -\bar{Q}_2^r$$

where $\bar{Q}_2^r$ denotes the per second discharge of the momentum of the gas issued through the nozzle. The equation of motion of the rocket on the projection on an axis, directed vertically upwards, is

$$m\ddot{x} = -mg + m_2 w_2^r + m_4 w_4^r + Q_2^r. \qquad (1.9)$$

we neglect the air resistance and pressure of the exit gases on the rocket.

Accuming that the average accelerations of particles are constant, i.e., the pump works with the uniform acceleration, we get

$$m_2 w_2^r + m_4 w_4^r =$$

where α = const, $\alpha > 0$.

we also have $m = m_o - \nu t - \gamma t^2$, ν= const >0, γ = const > 0.

therefore

$$v_2^r = bt + ct^2,$$

where b,c = const >0.

Hence

$$x = -g + \frac{a + bt + ct^2}{m_o - \nu t - \gamma t^2}. \qquad (1.10)$$

integrating (1.10), we get

$$x = x_o + x_o t - \frac{gt^2}{2} + \int_o^t dt \int_o^t \frac{a + bt + ct^2}{m_o - \nu t - \gamma t^2} dt. \qquad (1.11)$$

REMARKS: In obtaining the equation of motion of the rocket we have neglected the volume of nozzle, compared with the volume of the duct. We now suppose that it is possible to neglect the volume of the duct, compared with the volume of chamber and the jet propulsion nozzle. Consider another diagram. The fuel and the oxidiser fall in the combustion chamber. They get heated and rush to the nozzle and leave the rocket with an average velocity of $\bar{v}^r$. The restricted acceleration $\bar{w}_{rj}$ is responsible for any change in the velocity of J-th particle. In this case, the impulsive forces disappear and the Coriliso forces, as before, are equal to zero in view of the translatory motion of the rocket. We have then

$$\bar{R} = -\sum_j m_j \bar{w}_{rj} \qquad (1.12)$$

Denoting by $\bar{Q}^r$ the principal vector of an average momentum of particles which are found in the rocket at a given instant t,

$$\bar{Q}^r = \sum_j m_j \bar{u}_{rj} \qquad (1.13)$$

the summation is carried over all the particles found in the rocket jacket (casing or hull of the rocket) at a given instant t. At time $t + \Delta t$ the summation is taken over the set of the particles comprising the rocket

$$\bar{Q}^r (t + \Delta t) = \sum_{j_1} m_{j1} \bar{u}_{rj_1} (t + \Delta t)$$

then

$$\bar{Q}^r (t+\Delta t) = \sum_j m_j \bar{u}_{rj} (t+\Delta t) - \sum_{j_2} m_{j2} \bar{u}_{rj2} (t+\Delta t). \qquad (1.14)$$

The second summation on the right hand side of (1.14) extends over the set or the particles entering the composition of the rocket at instant t and leaving the rocket in interval Δt. From (1.13) we find the expression for an infinitely small increment

$$\Delta \bar{Q}^r = \bar{Q}^r (t + \Delta t) - \bar{Q}^r(t) =$$

$$= \sum_j m_j \bar{w}_{rj} \Delta t - \sum_{j_2} m_{j2} \bar{u}_{rj2} (t + \Delta t) + 0_2. \qquad (1.15)$$

divide (1.15) by Δt and proceeding to the limit as Δt tends to zero, we get

$$- \sum_j \bar{m}_j \bar{w}_{rj} = - \frac{d\bar{Q}^r}{dt} - \bar{Q}_1^r \qquad (1.16)$$

The quantity

$$\bar{Q}_2^r = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{j_2} \bar{m}_{j2} \bar{u}_{rj_2} (t + \Delta t)$$

represents per second discharge (expenditure) of average momentum of gas through the surface of the rocket. By virtue of (1.12) and (1.16) we get

$$\bar{R} = - \bar{Q}_2^r - \frac{d\bar{Q}^r}{dt}$$

If the process of combustion is steady then $\frac{d\bar{Q}^r}{dt} = 0$ and $\bar{R} = - \bar{Q}_2^r$

suppose the combustion takes place by uniformly increasing per second discharge of the mass, then the mass of the rocket may be expressed by formula (1.10) and $\frac{d\bar{Q}^r}{dt} = a$ = constant and ascending motion of the rocket will be described by the formula (1.11).

2. As a second example we consider a jet propelled aeroplane orbiting in a horizontal circle with an angular velocity of $\bar{\omega}$ and determines the radial projection of force F_r on its surface. Fig.2 explains the illustration. The point having a variable mass sucks air from the direction to which it is propelling with an average speed of v_1^r and issues out in the opposite direction with an average speed of

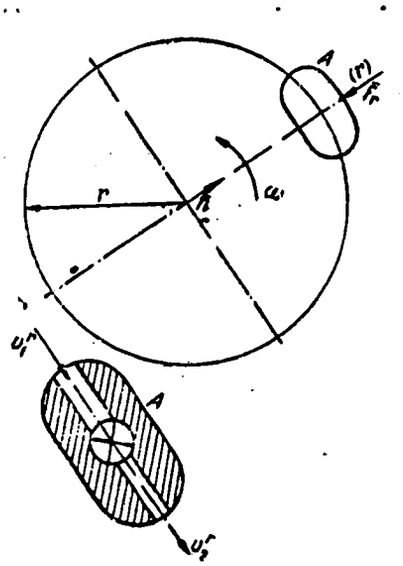

Рис. 2

Fig.2

v_2^r. In the process the point loses a part of its own mass and thus the lost particles fly out with the air at an average speed of v_2^r

Suppose the point describes a circle of radius R. Since the impulsive force and the average acceleration operate along the tangent to the circle, the equation (1.2) on projection on the radius vector gives

$$- mr\omega^2 = F_r + J_r,$$

where F_r^r is the projection of the force $\bar{F}$ on the radius vector, and the projection J_r of the Coriliso force is equal to $J_r = -2\omega(m_{\underline{BX}}v_1^r + m_{\underline{BUX}}v_2^r)$ where $m_{\underline{BX}}$ denotes the mass of the air present in the access duct, and $m_{\underline{BUX}}$ denotes the mass of air and of particles of the gas in the exit channel. The coordinate system connected with the aeroplane is taken

as the auxiliary system.

Then the required force is

$$F_r = - m \omega^2 r + 2 \omega (m_{BX} v_1^r + m_{BblX} v_2^r). \qquad (1.17)$$

$\frac{du_1}{dt}$ - per second expenditure of the mass of air through the cross-section of the access duct.

$\frac{du_2}{dt}$ - per second expenditure of the mass of gases through the cross section of the exit channel. l_1, l_2 - longitudinal dimensions of the access and the exit channels.

We will try to describe the motion of particles in the access duct. Let Δt be the time taken by the particles in passing from one section to another which are $\Delta \ell$ apart. The mass of the particles present in the canal between the above two sections is equal to $\frac{du_1}{dt} \Delta t$. Since $\Delta t = \frac{\Delta l}{v_1^r}$, then

$$m_{BX} v_1^r = \frac{du_1}{dt} l_1 \qquad (1.18)$$

Similarly we can find that

$$m_{BblX} v_2^r = \frac{du_2}{dt} l_2 \qquad (1.19)$$

substituting these values in (1.17), we have

$$F_r = - m \omega^2 r + 2\omega \left(\frac{du_1}{dt} l_1 + \frac{du_2}{dt} l_2 \right) \qquad (1.20).$$

For jet propelled aeroplane, we have

$$l_1 \ll r, \quad l_2 \ll r$$

therefore, the second component on the right hand side of (1.20) is much less in absolute value than the first component. The second component is referred to as the correction factor. If the aeroplane flies from a given instant with a shut - off motor, the second component on r.h.s. of (1.20) becomes zero and the radial force equals centrifugal force $m \omega^2 r$. It can be seen that the jet propelled plane on turning requires a bit less lateral force than the non-jet aeroplane.

2. MESHERSKY EQUATION:

1. Derivation of the Equation:

We will suppose that there is no movement of particles in the material point, i.e., the leaving particles remain in the matter point till they leave and the joining particles acquire the velocity of the point immediately after joining. This mode of change of mass is known as Meshersky mode. [53,55]

In this case

$$\bar{u}_r = \bar{w}_r = 0$$

therefore,

$$\bar{J} - \sum \bar{m}\bar{w}_r = 0$$

The equation of motion of point having a variable mass, it then becomes

$$m \frac{d\bar{v}}{dt} = \bar{r} + \bar{\Phi} \qquad (2.1)$$

the impulsive force $\bar{\Phi}$ is given by the formula (1.5). The final velocities $\bar{u}^2$ of the particles which are added to the matter point at a given instant t and the initial velocities $\bar{u}^1$ of the leaving particles are both equal to the velocity of the point. Therefore, we will have

$$\bar{\Phi} = \sum_{(1)} \frac{d\mu}{dt} (\bar{u} - \bar{v}) - \sum_{(2)} \frac{d\mu}{dt} (\bar{u} - \bar{v}) \qquad (2.2)$$

$\sum_{(1)}$ is carried over all the processes in which masses are added and

$\sum_{(2)}$ over all the processes in which masses are substracted.

Denote the velocities of the center of the inertia of joining and leaving the particles respectively by $\bar{u}_1$ and $\bar{u}_2$, then

$$\sum_{(1)} u d\mu = \bar{u}_1 dm_1, \quad dm_1 = \sum_{(1)} d\mu,$$

$$\sum_{(2)} \bar{u} d\mu = \bar{u}_2 dm_2, \quad dm_2 = \sum_{(2)} d\mu.$$

impulsive force can then be expressed as

$$\bar{\Phi} = \frac{dm_1}{dt}(\bar{u}_1 - \bar{v}) - \frac{dm_2}{dt}(\bar{u}_2 - \bar{v}) \qquad (2.3)$$

Since $\frac{dm_1}{dt}$, represents the rate of the addition of the mass and $\frac{dm_2}{dt}$ represents the rate of the substraction of the mass, therefore,

$$m = m_o + m_1 - m_2. \qquad (2.4)$$

m_o is constant and equal to the mass of the matter point at some initial instant t, m_1 and m_2 are non-negative functions characterizing the addition and substraction of the mass m_1 is the general mass of those particles which are added to the matter point from time t_o to t, and m_2 is the mass of those particles which leave the matter point at the same time.

Equation (2.1) can then be expressed as

$$m\frac{d\bar{v}}{dt} = \bar{F} + \frac{dm_1}{dt}(\bar{u}_1 - \bar{v}) - \frac{dm_2}{dt}(\bar{u}_2 - \bar{v}) \qquad (2.5)$$

It is known as Meshersky equation.

2. Special Cases of the Meshersky Equation:

(a) Suppose only the process of addition of mass takes place, then

$$m_2 = 0, \quad m = m_o + m_1$$

we get

$$\frac{dm_1}{dt} = \frac{dm}{dt}.$$

then the Meshersky equation will be written as :

$$m\frac{d\bar{v}}{dt} = \bar{F} + \frac{dm}{dt}(\bar{u} - \bar{v}) \qquad (2.6)$$

where $\bar{u}$ denotes the velocity $\bar{u}_1$ of the center of inertia of added particles.

(b) If only the process of the substraction of mass takes place then

$$m_1 = 0, \quad m = m_o - m_2.$$

therefore,

$$\frac{dm_2}{dt} = -\frac{dm}{dt}$$

the Meshersky equation (2.5) takes the form of (2.6) if the velocity $\bar{u}_2$ of the center of inertia of leaving particles is denoted by $\bar{u}$.

(c) Suppose the mass of the added particles in any time interval is equal to the mass of leaving particles in the same interval, then

$$m_1 = m_2, \quad m = m_o$$

the Meshersky equation (2.5) becomes

$$m\frac{d\bar{v}}{dt} = \bar{F} + \frac{dm_1}{dt}(\bar{u}_1 - \bar{u}_2)$$

(d) Consider the simultaneous addition and ejection of the particles when the velocities of the centers of inertia of joining and

leaving masses are equal. In this case

$$\bar{u}_1 = \bar{u}_2 = \bar{u}$$

$\therefore$ impulsive force is given by

$$\bar{\Phi} = \left(\frac{dm_1}{dt} - \frac{dm_2}{dt} \right) (\bar{u} - \bar{v}).$$

By (2.4) we get $\bar{\Phi} = \frac{dm}{dt} (\bar{u} - \bar{v})$. Here again Meshersky equation (2.5) takes the form (2.6).

(e) When the velocities of the changed masses $\bar{u}_1$ and $\bar{u}_2$ are equal to zero. In this case,

$$\bar{\Phi} = - \frac{dm}{dt} \bar{v}$$

then the Meshersky equation will be written as

$$\frac{d}{dt} (m\bar{v}) = \bar{F}.$$

(f) If the velocities of the changed masses are equal to the velocity of the point i.e., $\bar{u}_1 = \bar{u}_2 = \bar{v}$ then the impulsive force becomes zero and the Meshersky equation takes the form

$$m \frac{d\bar{v}}{dt} = \bar{F}.$$

Example showing the Inadequacy of the Meshersky Equation:

The Meshersky equation, as can be seen from its derivation, is strictly applied to the motion of point of a variable mass only when there is no interaction of particles. Thus if we wish to solve the problems by means of the Meshersky equation, then in the problem of ascending motion of the rocket we will not take into account that part of the reactive force caused by the relative acceleration of the particles, and in (the problem) studying the motion of the jet propelled aeroplane

We would consider an example which will clearly bring out the difference between the equation (1.2) and (2.5). A string BC passes over a fixed pulley (Fig.3). A weight Q is suspended at one end of the string and at the other end hangs a vessel of the constant transverse section weighing P; and filled with water weighing P_2. We will neglect

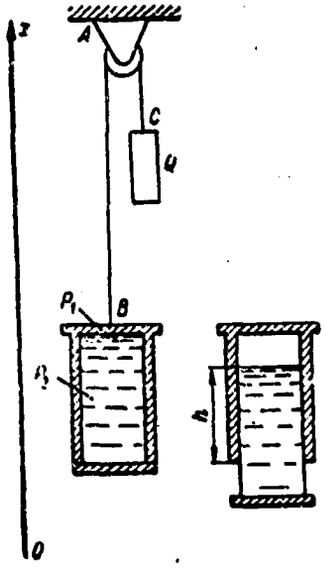

Рис. 3

Fig.3

the mass of the string and the pulley A. Suppose the system is in equilibrium. In this position, the shutter (damper) is quickly lowered (pulled down) on the vessel and the water begins to pour out. We will determine the acceleration of the weight Q.

Ignoring the friction of the water along the walls of the vessel, we may say that water will freely fall, i.e., it will have acceleration g. And the acceleration of the weight will be given by

$$w = g \frac{Q-P_1}{Q+P_1} \qquad (2.8)$$

We will apply the Meshersky equation to the matter point having a variable mass, consisting of the vessel, together with the water in it. In this case only the ejection of particles has relevance. Consequently we may proceed from equation (2.6). The velocity of matter point having variable mass in terms of the projection on X-axis will be $\dot{x}$, and the projection of the absolute velocity of leaving mass is equal to gt, therefore,

$$\frac{1}{g}\left(P_1 + P_2 \frac{h}{l} \right) \ddot{x} = T - P_1 - P_2 \frac{h}{l} + P_2 \frac{h}{gl} \left(- gt - \dot{x} \right).$$

Here $\frac{P_2}{g} \cdot \frac{h}{l}$ is the mass of the water in the vessel. $T = Q - \frac{Q}{g} \ddot{x}$ is the pull (tension) of the string. We have

$$\ddot{h} = - gt - \ddot{x}$$

therefore,

$$h = - \frac{gt^2}{2} - x + c$$

To determine the constant C we see that for t = 0, h = 1 and x =0, c = 1, therefore, the acceleration of the point is given by

$$\ddot{w} = \ddot{x} = \frac{g\left[Q - P_1 - \frac{P_2}{l}\left(1 - x - \frac{gt^2}{2}\right) + \frac{P_2}{gl}(\dot{x} + gt)^2 \right]}{P_1 + Q + \frac{P_2}{l}\left(1 - x - \frac{gt^2}{2}\right)} \qquad (2.9)$$

Equation (2.9) evidently does not agree with (2.8). Hence the Meshersky equation cannot be applied to this problem.

Now we apply the equation (1.2) to the matter point consisting of the vessel together with the water in it. We take the coordinate system associated (connected) with the vessel as the auxiliary system. They particles of the water dropping out from the vessel will have absolute accelerations and will be equal to the acceleration due to gravity g. The auxiliary system together with the vessel will have figurative acceleration $\ddot{x}$.

The projection of the average (relative) acceleration of the water particles is, therefore,

$$w_{rx} = -g - \ddot{x}.$$

The auxiliary system moves translationally and the velocities of the liquid particles change continuously without any impact. Consequently, in the present case,

$$\bar{J} = \bar{\Phi} = 0$$

Equation (1.2) in terms of projection on X-axis becomes:

$$\frac{1}{g}\left(P_1 + P_2\frac{h}{l}\right)\ddot{x} = T - P_1 - P_2\frac{h}{l} + R_x,$$

where projection of the reactive force is $R_x = -\frac{P_2 h}{gl} w_{rx} = \frac{P_2 h}{gl}(g + \ddot{x})$, therefore,

$$\frac{1}{g} P_1 \ddot{x} = T - P_1.$$

In view of the relation

$$T = Q - \frac{Q}{g}\ddot{x},$$

we have

$$w = \ddot{x} = g\frac{Q - P_1}{Q + P_1},$$

which is the same as (2.8).

3. TWO PROBLEMS OF SIALOKOV3.

1. First Problem of Sialokov:

We discuss the motion of point having variable mass and emitted (radiated) in the cosmic space. We will neglect the inner motion (interaction) of the particles and assume that the reactive force considerably exceeds the forces of resistance and gravity. We write the relative velocity of the radiated particles as $\bar{u}_r = \bar{u} - \bar{v} = -\bar{u}_r\bar{I}$, where $\bar{I}$ is the unit vector. Given that u_r = constant, find the increment in the velocity of the point when a specified change in its mass occurs. This enunciates the first problem of Sialokova (104) (1930).

The studied motion of the point is applicable to the case in the preceding section and is determined by the equation(2.6) which takes the form of

$$m \frac{d\bar{v}}{dt} = - \frac{dm}{dt} u_r \bar{i}.$$

This gives

$$d\bar{v} = - u_r \frac{dm}{m} \bar{i} \qquad (3.1)$$

Suppose the reactive force does not change its direction in the space, then $\bar{i}$ = constant. Integrating (3.1), we get

$$\bar{v} = \bar{v}_0 + u_r \ln \frac{m_0}{m} \bar{i} \qquad (3.2)$$

where $\bar{v}_0$ is the initial velocity of the point and m_0 its initial mass. m denotes the mass of the point at instant t when its velocity is calculated to be $\bar{v}$.

(3.2) is called the Sialokov's formula. The Sialokov formula shows that the increment in the velocity of the point having a variable mass under the Sialokov's conditions is proportional to the logauthmic ratio of the initial and first masses.

2. <u>Second Problem of Sialokov:</u>

Sialokov published his research[104] about the motion of the rocket in the constant field of gravity. We will find the solution of the second problem of Sialokov in the modern notations.

Let the point move vertically upward in the uniform (homogeneous) field of gravity and its mass vary according to the law

$$m = m_0 e^{-\alpha t}, \quad m_0 = \text{const}, \quad \alpha = \text{const} > 0. \qquad (3.3)$$

Assuming that the particles are propelled with uniform speed, investigate the velocity and the law of motion of the point.

Equation (2.6) becomes

$$m \frac{d\bar{v}}{dt} = - m\bar{g} - \frac{dm}{dt} \bar{u}_r \qquad (3.4)$$

Projecting both the sides of the equation (3.4) on the vertical axis, we get

$$m \frac{dv}{dt} = - mg - \frac{dm}{dt} u_r \qquad (3.5)$$

From (3.3) it follows that

$$\frac{dm}{m} = \alpha dt$$

therefore, we may rewrite equation (3.5) as

$$\frac{dv}{dt} = g (q - 1) \qquad (3.6)$$

where $q = \frac{\alpha v_r}{g}$ is integration of (3.6) gives the rule of change of velocity of the point, namely,

$$v = (q - 1) gt + v_0$$

Putting $v = \frac{ds}{dt}$, we find the law of motion of a point in the form

$$S = (q - 1) \frac{gt^2}{2} + v_o t. \qquad (3.7)$$

It indicates that the initial reading of the distance S moved by the point coincides with its initial position.

Suppose the fuel burns for time t which we will now calculate. From (3.3) we will have

$$m = m_o e^{-\alpha t_1}$$

therefore,

$$t_1 = \frac{\beta}{\alpha} \qquad (3.8)$$

where

$$\beta = \log \frac{m_o}{m}$$

When the active motion ceases, the velocity of the point will be

$$v_1 = (q - 1) g \frac{q\beta}{2\alpha^2} + v_o$$

The displacement of the point during the active motion is found by substituting the value of t, in (3.7)

$$S_1 = (q - 1) \frac{g\beta^2}{2\alpha^2} + v_o \frac{\beta}{\alpha} \qquad (3.9)$$

We will now develop a formula for the full distance traversed by the point during the active motion and also the distance moved during the free flight till it comes to a stand still.

Let S_2 be the sector length moved by the point after the fuel has completely burnt. Then by the law of transformation of the Kinetic energy of the point with a constant mass, we write

$$-\frac{mv_1^2}{2} = -mgs_2.$$

This gives

$$S_2 = \frac{v_1^2}{2g} = \frac{1}{2g}\left[(q-1)\frac{g\beta}{\alpha} + v_0\right]^2 \qquad (3.10)$$

The total distance traversed by the point is found from (3.9) and (3.10), i.e.,

$$S = S_1+S_2 = (q-1)\frac{g\beta^2}{2\alpha^2} + v_0\frac{\beta}{\alpha} + \frac{1}{2g}\left[(q-1)\frac{g\beta}{\alpha} + v_0\right]^2 \qquad (3.11)$$

The investigation of external case is of special interest. For this we substitute the value of q in terms of α from the formula

$$q = \alpha g^{-1} v_r$$

and consider the distance as the function of the argument α. Equation (3.9) will then be replaced by

$$S_1 = \frac{v_0\beta}{\alpha} + (\alpha v_r - g)\frac{\beta^2}{2\alpha^2}$$

Differentiating the above w.r.t α and equating it to zero we get

$$\alpha = \frac{2g\beta}{2v_0 + \beta v_2}$$

It can be seen that this value of α maximises the function S_1

$$S_{1\ max} = \frac{(2v_o + \beta v_r)^2}{8g}$$

substituting the value of q in terms of α in (3.11) it becomes

$$S = \frac{v_o \beta}{\alpha} + (\alpha v_r - g)\frac{\beta^2}{2\alpha^2} + \frac{1}{2g}\left[(v_o + (\alpha v_r - g)\frac{\beta}{\alpha}\right]^2 ,$$

Similarly the maximum altitude of the rocket will be worked out for the value of as obtained from the equation

$$\frac{1}{2}\frac{\beta^2 v_r}{\alpha^2} = 0.$$

i.e., as $\alpha \longrightarrow \infty$. In this case

$$S_{max} = \frac{1}{2g}(v_o + v_r \beta)^2$$

Since the time of active combustion is given by $t_1 = \frac{\beta}{\alpha}$ for achieving S_{max} in the uniform field of the gravity disregarding resistance the supply of the fuel should be burned as quickly as possible.

4. MOTION OF SPHERICAL DROP ON EVAPORATION OR CONDENSATION OF THE VAPORS ON ITS SURFACE:

1. Formulation of Problems:

Studying the motion of the spherical shaped water drop in the stationary atmosphere, saturated with water vapors Sommerfield[23] suggests to proceed from the assumption that on condensation the increase in the mass of the drop is proportional to the area of its

surface. However, this assumption is justified only in the case of evaporation under vacuum.

The low speed motions in a gaseous medium like the rate of condensation and evaporation is determined by the Maxwell's law

$$\frac{dm}{dt} = ekr \qquad (4.1)$$

where m denotes the mass of the drop $e = \pm 1$ ($+ V_e$ sign is taken for condensation, and $- v_e$ sign for evaporation); k - a positive constant depending on the diffusion coefficient, difference in concentration of vapor on the surface of the drop and the distance from it, and r is the radius of the drop.

In the range of 100 to 1000 of R_e, the experiments of Freshinga[109] and his successors[103] demonstrated that the rate of condensation and evaporation is given by the formula

$$\frac{dm}{dt} = epr \sqrt{rv}, \qquad (4.2)$$

where p = constant > 0, v - velocity of the drop . Formula (4.2) is corroborated by the theory of the boundary layer[103].

Below is given the discussion of two problems:

In section 2 slow and a comparatively rapid fall of a drop in motionless gaseous medium is discussed when condensation or evaporation of the vapors occurs on its surface. In section 3 the horizontal motion of the liquid fuel drop in respect of the gas flow is discussed. Since the particles of the vapor on condensation of (after evaporation)

have zero velocity, the reactive force R, arising from the change of the mass equals $\frac{dm}{dt}\bar{v}$.

2. The Problem of Free and Slow Fall of a Spherical Drop on whose Surface Evaporation or Condensation Occurs (71)

For slow falling of the drop the medium resistance is taken to be $-\alpha\pi r^2 \bar{v}$, where α = const. > 0. Projecting on the vertical axis x, directed downwards, the Meshersky equation in the present case yields.

$$m\ddot{x} = mg - (\alpha\pi r^2 + ekr)\,x. \qquad (4.3)$$

We have

$$m = \frac{4}{3}\gamma\pi r^3 \qquad (4.4)$$

Where γ is the density. Differentiating (4.4) in respect of time and equating the resulting derivative to (4.1), we get

$$r = \sqrt{r_0^2 + \nu t}, \qquad \nu = \frac{ek}{2\pi\gamma} \qquad (4.5)$$

Using (4.4) and (4.5), equation (4.3) can be written as

$$\ddot{x} + \frac{3}{2}\left[\frac{\alpha}{2\gamma\sqrt{r_0^2 + \nu t}} + \frac{\nu}{r_0^2 + \nu t}\right]\dot{x} = g. \qquad (4.6)$$

If we disregard the medium resistance, i.e., $(\alpha = 0,)$ then integrating the equation (4.6) and using (4.5), we get

$$x = \frac{1}{r^3}\left[\dot{x}_0 r_0^3 - \frac{2}{5}\,\frac{g}{\nu}\,(r_0^5 - r^5)\right] \qquad (4.7)$$

Integrating the linear equation (4.6) when $\alpha \neq 0$, we get

$$x = \frac{C_1}{(r_o^2+\nu t)^{\frac{3}{2}} \cdot q\sqrt{r_o^2+\nu t}} + \frac{4g\gamma}{3\alpha}\left[\sqrt{r_o^2+\nu t} - \frac{4}{q} + \right.$$

$$\left. + \frac{12}{q^2\sqrt{r_o^2+\nu t}} - \frac{24}{q^3(r_o^2+\nu t)} + \frac{24}{q^4(r_o^2+\nu t)^{\frac{3}{2}}}\right] \qquad (4.8)$$

Constant of the solution (4.8) is given by

$$C_1 = \left[x_o - \frac{4g\gamma}{3\alpha} r_o - \frac{4}{q} + \frac{12}{q^2 r_o^2} + \frac{24}{q^4 r_o^3}\right] r_o^3 \cdot e^{qr_o} \qquad (4.9)$$

We now consider the case of little evaporation and little condensation and restrict the solution (4.8) to those terms which contain the parameter ν in the first order. Formula (4.9) then yields

$$x = \frac{4g\gamma r_o}{3\alpha} - \left(\frac{4g\gamma r_o}{3\alpha} - x_o\right) \cdot e^{-\frac{3\alpha t}{4\gamma r_o}} + \nu\left\{\frac{4g\gamma}{3\alpha}\left(\frac{t}{2r_o} - \frac{8\gamma}{3\alpha}\right) + \right.$$

$$\left. + \left[-\frac{3}{16r_o^2}\left(\frac{4g\gamma r_o}{3\alpha} - x_o\right)\left(\frac{\alpha t^2}{\gamma r_o} - 8t\right) + \frac{32g\gamma^2}{9\alpha^2} \cdot e^{-\frac{3}{4}\frac{t}{r_o}}\right.\right. \qquad (4.10)$$

In the case under consideration formula (4.5) becomes

$$r = r_o + \frac{1}{2}\frac{\nu t}{r_o}$$

therefore, neglecting the exponential factors, we get

$$x = \frac{4g\gamma r}{3\alpha} - \frac{32\nu g\gamma^2}{9\alpha^2} \qquad (4.11)$$

Formula (4.11) gives the critical (limiting) velocity of the drop, acquired by it after a prolonged interval of time. It further shows that on condensation the critical (limiting) velocity diminishes by $\frac{16\lambda g}{9\pi x^2}$ and on evaporation it increases by the same amount.

By integrating equation (4.7) we get

$$x = x_o + \frac{2x_o r_o^2}{\nu}\left(1 - \frac{r_o}{r}\right) - \frac{g r_o^4}{\nu 2}\left[1 - \frac{4}{5}\frac{r_o}{r} - \frac{1}{5}\left(\frac{r}{r_o}\right)^4\right] \qquad (4.12)$$

Formula (4.12) expresses the law of falling of a drop when the medium resistance is disregarded.

When $\lambda \neq 0$ integration of equation (4.8) yields

$$x = C_2 - \frac{2C_1}{\nu}\left[e^{-qr}\frac{1}{r} - qE_1(-qr)\right] +$$

$$+ \frac{8g\lambda}{3\nu x}\left(\frac{r^3}{3} - \frac{2t}{q} + \frac{12r}{q^2} - \frac{24\ln r}{q^3} - \frac{24}{q^4 r}\right), \qquad (4.13)$$

in which the constant C_2 is given by

$$C_2 = x_o + \frac{2C_1}{\nu}\left[e^{-qr_o}\frac{1}{r_o} - qE_1(-qr_o)\right] -$$

$$- \frac{8g\lambda}{3\nu x}\left(\frac{r_o^3}{3} + \frac{12r_o}{q^2} - \frac{24\ln r_o}{q^3} - \frac{24}{q^4 r_o}\right) \qquad (4.14)$$

In equation (4.13), the radius r of thedrop is obtained from the equation (4.5) and in the formulae (4.13) and (4.14), $E_1(x)$ denotes the integral exponential function $\int_{-\infty}^{x} \frac{e^y}{y}\,dy$.

In case of insufficient condensation or evaporation, integrating the equation (4.10), we get

$$x = x_o + \frac{4gr_o}{3\alpha}\left(\frac{4g\gamma r_o}{3\alpha} - x_o\right)\left(1-e^{-\frac{3\alpha t}{4\gamma r_o}}\right) + \frac{4g\gamma r_o t}{3\alpha} +$$

$$+\sqrt{\left[\frac{4g}{3\alpha}\left(\frac{t^2}{4r_o} - \frac{3\gamma t}{3\alpha}\right) - \frac{1}{5}\left(\frac{4g\gamma r_o}{3\alpha} - x_o\right)\left(\frac{16\gamma r_o t}{3\alpha} - t^2\right)\right] e^{-\frac{3\alpha t}{4\gamma r_o}} +}$$

$$+ \frac{128g\gamma^3 r_o}{27\alpha^3}\left(1 - e^{-\frac{3\alpha t}{4\gamma r_o}}\right)\Bigg] \qquad (4.15)$$

For large intervals of time, the exponential factors in the formula (4.15) may be neglected. Thus we get

$$x = x_o + \frac{4gr_o}{3}\,\frac{4g\ r_o}{3} - x_o + \frac{4g\ r_o^{\,t}}{3} +$$

$$+ \frac{4g}{3}\,\frac{t^2}{4r_o} - \frac{8\ t}{3} + \frac{128g\ ^3r_o}{27\ \ 3} \qquad (4.16)$$

We note that the derivative of (4.16) agrees with formula (4.11).

3. Problem of Free Rapid fall of the Spherical drop on whose Surface Evaporation or Condensation Occurs[71]

When the drop falls rapidly we take the medium resistance to be $\beta\pi r^2 v^2$. Projecting on the vertical axis x and applying formula (4.2) we get

$$\ddot{x} = g - \frac{3}{4}\frac{\beta x^2}{\gamma r} - \frac{3}{4}\,\frac{\rho}{\pi\gamma r^{\frac{3}{2}}}\,x^{\frac{3}{2}} \qquad (4.17)$$

Differentiating the equation (4.4) and equating it to the left hand side of equation (4.2), we will have

$$r = \frac{\rho}{4\pi\gamma}\sqrt{\frac{x}{r}} \qquad (4.18)$$

We will examine the case when P is small and find out the solutions of the equations (4.17) and (4.18) in series, $\mu = \frac{\rho}{4\pi\gamma}$. To make it simple we allow the solution to retain only those terms which contain the parameter μ of the first order. Hence the solution is obtained in the form $r = \rho + \mu\xi$, $x = v + \mu$. In order to determine zero terms we get the equations

$$v = g - \frac{3}{4}\frac{\beta}{\rho\gamma}v^2, \quad \rho = 0$$

From here it follows that

$$\rho = r_0, \quad v = v_1 \frac{v_0 + v_1 \operatorname{th}\frac{gt}{v_1}}{v_1 + v_0 \operatorname{th}\frac{gt}{v_1}}, \quad v_1 = 2\sqrt{\frac{gr_0\gamma}{3\beta}} \qquad (4.19)$$

The variables ξ and η are determined from the zero initial features (conditions) of the following equations

$$\eta = -\frac{3}{2}\frac{\beta v}{\gamma r_0}\eta + \frac{3\beta}{4\gamma}\left(\frac{v}{r_0}\right)^2\xi - 3\left(\frac{v}{r_0}\right)^{\frac{3}{2}}, \quad \xi = \sqrt{\frac{v}{r_0}}. \qquad (4.20)$$

Equation (4.20) yields the values of ξ and η in squares.

Formula (4.19) shows that when the motion continues for a sufficiently long time the quantity v approximates to v_1. Since the

bodies falling in a resisting medium the critical velocities are attained fairly rapidly, we take that beginning from a certain moment of time v equals v_1.

$$\eta + \frac{3\beta v_1}{2\gamma r_o} \quad \eta = \frac{3\beta}{4\gamma}\left(\frac{v}{r_o}\right)^2\zeta - 3\left(\frac{v^1}{r_o}\right)^{\frac{3}{2}}, \quad \xi = \sqrt{\frac{v_1}{r_o}}\, t. \qquad (4.21)$$

the integrating equation (4.20), we have

$$\eta = \sqrt{\frac{v_1}{r_o}}\left[\frac{t}{2}\sqrt{\frac{v_1}{r_o}} - \frac{7\gamma}{3\beta}\left(1 - e^{-\frac{3\beta v_1}{2\gamma r_o}t}\right)\right] \qquad (4.22)$$

Formula (4.22) and the expression $r = r_o + \mu\xi$ yields

$$x = 2\sqrt{\frac{gr\gamma}{3\beta}} - \frac{7\,ep}{12\pi\beta}\sqrt{\frac{v_1}{r_o}}\left(1 - e^{-\frac{3\beta v_1}{2\gamma r_o}t}\right)$$

The first part of the above result represents the critical velocity of the drop of radius r. The second part of the result indicates that the critical velocity decreases on condensation and increases on evaporation. If we discard the exponential factor, the change in critical velocity is

$$\frac{7\rho}{6\pi\beta}\sqrt{\frac{g\gamma}{3\beta}}$$

4. Change in the Radius of a Spherical Drop on a Horizontal Flight:

Working on the processes of combustion two things are important to know: the motion of the liquid fuel drop in respect of a horizontal gas flow and the change inthe radius of the drop as a result of evaporation: In practical problems the actual values of the initial

velocity r of the coefficient of the Kinemeatic viscosity μ are such that the Reonald numbers do not exceed 1000.

In the works of Putnum ([81]) a formula for determining the resistance coefficient C_D of the spherical drop when $Re < 1000$ has been suggested as

$$\frac{1}{24} C_D Re = 1 + \frac{1}{8} Re^{\frac{2}{3}} \qquad (4.23)$$

According to Putmum formula (4.23) gives good results.

We will use formula (4.2) for the computation of the change of mass. As has been described in section 1 of the present paragraph the formula holds for Re ranging from 100 to 1000. In the present case the constant e in formula (4.2) is equal to - 1, p is some positive constant and mass is determined by formula (4.4) in which γ denotes the density and r is the radius of the drop. Setting the projection of the differential equation (of motion) in the direction of the horizontal motion of the drop, we proceed from the Meshersky equation (2.5) in which reactive force $R = \frac{dm}{dt} \bar{v}$.

Take

$$\tau = (ae^{bt} - c^2)^{\frac{1}{4}}, \qquad (4.24)$$

$$a = 1 + c^2, \; c^2 = \frac{1}{6} Re_H^{\frac{3}{2}}, \; b = 3\gamma^{-1} \mu r_H^{-2} \qquad (4.25)$$

where t - time; Re - Reonald number at the beginning of the motion; r_H - initial radius of the drop. Initial value of τ when t = 0 is unity.

The formula for the velocity of the drop obtained on the basis of (4.23) by putting p = 0 in the result (4.2) may be transformed as

$$v = v_H \tau^{-6} \tag{4.26}$$

where v_H - initial velocity of the drop,

Retaining the terms of first order smallness (infinitesmial) in respect of the coefficient p, we find the shrinkage (decrement) in the radius of the drop.

In formula (4.2) put

$$r = r_H + p\xi \tag{4.27}$$

then it follows

$$\xi = -\frac{1}{4\pi} \gamma^{-1} r_H^{-1/2} v^{1/2} \tag{4.28}$$

Proceeding to the independent variable τ . (i.e. differentiating w.r.t.) and using equations (4.24), (4.26) and (4.28), we get

$$\frac{d\xi}{d\tau} = -4A (C^2 + \tau^4)^{-1} \tag{4.29}$$

where

$$A = \frac{1}{4\pi} \gamma^{-1} r_H^{-1/2} v_H^{1/2} b^{-1} \tag{4.30}$$

integrating equation (4.29) and taking the initial values $\tau_H = 1$, $\xi_H = 0$

$$\xi = -\frac{1}{\sqrt{2}} AC^{-\frac{3}{2}} \left[\ln \frac{(c+\tau\sqrt{2c}+\tau^2)(c-\sqrt{2c}+1)}{(c-\tau\sqrt{2c}+\tau^2)(c+\sqrt{2c}+1)} + \right. \tag{4.31}$$

$$\left. + 2 \operatorname{arctg} \frac{\sqrt{2c}\tau}{c-\tau^2} - 2 \operatorname{arctg} \frac{\sqrt{2c}}{c-1} \right]$$

Derivative with respect to τ of the function

$$(\tau) = \ln \frac{c+\tau\sqrt{2c}+\tau^2}{c-\tau\sqrt{2c}+\tau^2}$$

is greater than zero for $\tau < \sqrt{c}$ and less than zero for $\tau > \sqrt{c}$. Therefore, the first term in the square bracket of (4.31), representing the difference $f(\tau) - f(1)$, is positive and increasing for $1 < \tau < \sqrt{c}$, and when $\tau > \sqrt{c}$ it decreases and tends to become negative for sufficiently large values of τ .

The second term in the square bracket is greater than zero and increases for τ lying between 1 and $\sqrt{c}$. At a point $\tau = \sqrt{c}$, this term suffers discontinuity of the first kind and then again increases.

As regards the physical meaning, ξ must be continuous and monotonic decreasing. In order to preserve the continuity and the monotonocity of the variable ξ after passing through the point of discontinuity $\tau = \sqrt{c}$ we would add 2 for values $\tau > \sqrt{c}$.

Making $\tau \to \infty$ and using equation (4.27), the shrinkage (decrement) Δ r in the radius of the drop till it stops can be expressed as

$$\Delta r = -\frac{1}{\sqrt{2}}\, pAc^{-\frac{3}{2}} \left(2\pi - \ln \frac{c+\sqrt{2c}+1}{c-\sqrt{2c}+1} - 2\,\mathrm{arctg}\,\frac{\sqrt{2c}}{c-1}\right) \qquad (4.32)$$

Using (4.31) and with the help of the Meshersky equation of a point having a variable mass for the spherical drop, we may obtain the formula in the form of quadrature giving the extension in a horizontal distance as a result of evaporation.

5. Some Questions of Mechanics of a Point Having Variable Mass as Examined by Meshersky:

The following questions were put forth. Finding the reaction of the ideal (couplings) constraints, converse problems of the dynamics of a point having variable mass, motion of a point having variable mass under the action of central forces and nature of dependence of varying mass in the velocity.

The accuracy of the resultsof Meshersky (53,55) is shown.

1. Reaction of the Ideal Constraints Experienced by the point Whose Mass Changes Without Impact:

When the velocities of the varying mass u_1 and u_2 are equal to the velocity v of the point having variable mass, the point does not experience impacts and the impulsive forces will then be zero. The Meshersky equation (2.5) becomes

$$m\frac{\bar{dv}}{dt} = \bar{F},\ \frac{\bar{dv}}{dt} = \frac{1}{m}\ \bar{F}. \qquad (5.1)$$

Accordingly, in terms of projections, we find

$$\ddot{x} = X_1,\ \ \ddot{y} = Y_1,\ \ \ddot{z} = Z_1, \qquad (5.2)$$

where the projections of the force are carried to the unit mass.

$$X_1 = \frac{1}{m} X,\ Y_1 = \frac{1}{m} Y,\ Z_1 = \frac{1}{m} Z$$

Equation (5.1) and (5.2) show that the free point having a variable mass which does not experience impact during the variation of the mass, moves under the action of the given force in the same manner as the free point of constant unit mass moves under the action of the force calculated on unit mass and under the same initial conditions.

Let the motion of the point having variable mass be subjected to a constraint of smooth surface.

$$\varphi (x, y, z, t) = 0 \qquad (5.3)$$

Combining with the active forces the reaction of the surface

$$\lambda \operatorname{grad} \varphi ,$$

leads us to the case of a free point having a variable mass. Therefore, using (5.1) and (5.2) we get

$$m \frac{dv}{dt} = F + \lambda \operatorname{grad} \varphi \ . \qquad (5.4)$$

$\lambda_1 = m^{-1}\lambda$, then we have

$$\ddot{x} = X_1 + \lambda_1 \frac{\partial \varphi}{\partial x} ,\ \ddot{y} = Y_1 + \lambda_1 \frac{\partial \varphi}{\partial y} ,\ z = Z_1 + \lambda_1 \frac{\partial \varphi}{\partial z} \qquad (5.5)$$

Where λ_1 is determined from the constraint equation (5.3) If the mass of the point depends on the distance, then we may also take into account the equation

$$\frac{ds}{dt} = \sqrt{x^2 + y^2 + z^2}. \tag{5.6}$$

From (5.5) we conclude that the motion of constraint point having variable mass can be construed as the motion of point having constant unit mass.

2. Reaction of Ideal Constraints Experienced by the Point, Whose change of Mass is Accompanied by an Impact:

Let the motion of a point having a variable mass and which experiences impacts during the change in its mass be subjected to an ideal constraint. Since the surface (5.3) is supposed to be smooth, its reaction μ grad acts in the direction of the normal and vector equation (2.5) can be expressed as

$$m\frac{\bar{dv}}{dt} = \bar{F} + \frac{dm_1}{dt}(\bar{u}_1 - \bar{v}) - \frac{dm_2}{dt}(\bar{u}_2 - \bar{v}) + \mu \text{grad}\varphi . \tag{5.7}$$

Denoting the projections of U_1 and U_2 on the axis of the coordinate system by $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$, we get

$$\begin{aligned} mx &= X + \frac{dm_1}{dt}(\alpha_1 - \dot{x}) - \frac{dm_2}{dt}(\alpha_2 - \dot{x}) + \mu\frac{\partial\varphi}{\partial x}, \\ my &= Y + \frac{dm_1}{dt}(\beta_1 - \dot{y}) - \frac{dm_2}{dt}(\beta_2 - \dot{y}) + \mu\frac{\partial\varphi}{\partial y}, \\ mz &= Z + \frac{dm_1}{dt}(\gamma_1 - \dot{z}) - \frac{dm_2}{dt}(\gamma_2 - \dot{z}) + \mu\frac{\partial\varphi}{\partial z} . \end{aligned} \tag{5.8}$$

Multiplier μ will be determined from the constraint equation (5.3). Differentiating (5.3) we have:

$$\frac{\partial\varphi}{\partial x}\dot{x} + \frac{\partial\varphi}{\partial y}\dot{y} + \frac{\partial\varphi}{\partial z}\dot{z} + \frac{\partial\varphi}{\partial t} = 0, \qquad (5.9)$$

$$\frac{\partial\varphi}{\partial x}\ddot{x} + \frac{\partial\varphi}{\partial y}\ddot{y} + \frac{\partial\varphi}{\partial z}\ddot{z} =$$

$$= -\left(\dot{x}\frac{d}{dt}\frac{\partial\varphi}{\partial x} + \dot{y}\frac{d}{dt}\frac{\partial\varphi}{\partial y} + \dot{z}\frac{d}{dt}\frac{\partial}{\partial z} + \frac{d}{dt}\frac{\partial}{\partial t}\right). \qquad (5.10)$$

putting in (5.10) x, y, z as obtained from (5.8) and using (5.9), we get:

$$\mu = \lambda + \frac{dm_1}{dt}\tilde{\Phi}(x_1, \beta_1, \gamma_1) - \frac{dm_2}{dt}\tilde{\Phi}(x_2, \beta_2, \gamma_2). \qquad (5.11)$$

In(5.11)

$$\lambda = -\frac{1}{(\Delta\varphi)^2}\left[\frac{\partial\varphi}{\partial x}X + \frac{\partial\varphi}{\partial y}Y + \frac{\partial\varphi}{\partial z} + \right.$$

$$\left. + m\left(\dot{x}\frac{d}{dt}\frac{\partial\varphi}{\partial x} + \dot{y}\frac{d}{dt}\frac{\partial\varphi}{\partial y} + \dot{z}\frac{d}{dt}\frac{\partial\varphi}{\partial z} + \frac{d}{dt}\frac{\partial\varphi}{\partial t}\right)\right], \qquad (5.12)$$

$$\Delta\varphi = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial t}\right)^2},$$

$$\tilde{\Phi}(x_i, \beta_i, \gamma_i) = -\frac{1}{(\Delta\varphi)^2}\left(\frac{\partial\varphi}{\partial x}_1 + \frac{\partial\varphi}{\partial y}\beta_i + \frac{\partial}{\partial z}\gamma_i + \frac{\partial\varphi}{\partial t}\right) \quad (i=1, 2)..(5.13)$$

Assume

$$\alpha_1 = x_2 = x, \quad \beta_1 = \beta_2 = y, \quad \gamma_1 = \gamma_2 = z.$$

then as a result of (5.9) we get

$$\widetilde{\Phi}(\alpha_i, \beta_i, \gamma_i) = 0 \quad (i = 1, 2)$$

and by (5.11)

$$\mu = \lambda$$

Therefore, λ is called the reaction multiplier (factor) brought into play by the motion of the point and the action of the force F. From here we must conclude that the component reaction

$$\left[\frac{dm_1}{dt}\widetilde{\Phi}(x_1, \beta_1, \gamma_1) - \frac{dm_2}{dt}\widetilde{\Phi}(x_2, \beta_2, \gamma_2)\right] \operatorname{grad}\varphi \qquad (5.14)$$

is caused by the impact process of the change of mass and it will be referred to as impact component reaction. If the changing masses move in the surface (5.3), then

$$\Phi(\alpha_i, \beta_i, \gamma_i) = 0 \quad (i = 1, 2)$$

and the impact component reaction does not arise. When only the addition or ejection of particles gives rise to the exclusion of the other, the impact component reaction can be expressed by

$$\frac{dm}{dt}\Phi(\alpha, \beta, \gamma) \operatorname{grad}\varphi.$$

Let the point move on the curve of the intersection of the two surfaces.

$$\varphi_1 (x, y, z, t) = 0, \varphi_2 (x, y, z, t) = 0.$$

In this case, the reaction of the curve is equal to the geometrical sum of the reactions of the surfaces

$$\mu_1 \operatorname{grad} \varphi_1 + \mu_2 \operatorname{grad} \varphi_2.$$

whereas μ_1, μ_2 can be determined by the method outlined above. If the surfaces are orthogonal to each other, then μ_1 and μ_2 are expressed by the equation (5.11).

3. Converse Problems of Dybamics of a Point Having Variable Mass.

I.V. Mrdhrtdky in hid msdyrt's dissertation (53), (1897) discusses some converse problems of the mechanics of a variable mass, in which it is required to find out the law of change of mass of a point when its motion under given forces is known. We examine another problem connected with it.

Let the velocities of the changing masses be equal to the velocity of the point and let the point describe a plane curve

$$y = f(x) \qquad (5.15'$$

under the influence of the medium resistance and the other forces, resultant of which calculated for a unit mass, depends only on the

Denote the projections of the resultant by X and Y and let W denote the medium resistance, calculated per unit mass divided by the velocity of the point. Then equation (5.2) becomes

$$\ddot{x} = X - W\dot{x}, \quad \ddot{y} = Y - W\dot{y}. \tag{5.16}$$

As the equation of the curve is $y = f(x)$, differentiating w.r.t. we have

$$\ddot{y} = f'\ddot{x} + f''\dot{x}^2,$$

where f' is the derivative of f w.r.t. x.

Therefore (5.16) now becomes

$$Y - W\dot{y} = f'\ddot{x} - f'W\dot{x} + f''\dot{x}^2$$

Since

$$\dot{y} = f'\dot{x}, \tag{5.17}$$

we find

$$\dot{x}^2 = \frac{Y - f'X}{f''} \tag{5.18}$$

$$\dot{x} = e\sqrt{\frac{1}{f''}(Y - f'X)}, \quad e = \pm 1. \tag{5.19}$$

Differentiating (5.18) w.r.t. (5.16), (5.17) and (5.19) yield

$$W = \frac{e}{2}\sqrt{\frac{f''}{Y - f'X}}\,G \tag{5.20}$$

Where

$$G = 3X - \frac{1}{f'}\left[Y_x + f'Y_y - f'(X_x + f'X_y)\right] + \frac{f''}{f'^2}(Y - f'X) \qquad (5.21)$$

in which X_x, X_y, Y_x, Y_y, are the corresponding partial derivatives of X and Y.

Equation (5.17) and (5.19) provide the velocity of the point as

$$v = \sqrt{\frac{1}{f''}(1 + f'^2)(Y - f'X)} \qquad (5.22)$$

therefore, the medium resistance is given by

$$W_v = \frac{e}{2}G\sqrt{1 + f'^2} \qquad (5.23)$$

When a heavy point having variable mass describes the curve (5.15) in the vertical plane, we have

$$X = 0,\ Y = g,\ G = \frac{f''}{f'^2}g$$

therefore,

$$W_v = \frac{eg}{2}\,\frac{f''}{f'^2}\sqrt{1 + f'^2}$$

mass of the point must be equal to the real resistance divided by W_v.

Now suppose the velocities of the changing masses are equalto zero, then the motion of the point in the plane is described by the equations

$$m\ddot{x} = mX - (\check{W} + \frac{dm}{dt})\,\dot{x}, \qquad (5.24)$$

$$m\ddot{y} = mY - (\check{W} + \frac{dm}{dt})\,\dot{y},$$

where $\check{W}$ is the medium resistance divided by the velocity of the point.

Taking

$$W = \frac{1}{m}\,(\check{W} + \frac{dm}{dt}),$$

we transform the equations (5.24) to the form (5.16) and the law of change of mass along the trajectory of the motion performed under the influence of given forces is thus formulated.

Suppose only the processes of addition or ejection of particles to the exclusion of others takes places and the velocity of the changing mass is directed towards the velocity of the point. Then the vector equation (2.6) on projections on the axes OX and OY will becomes

$$m\ddot{x} = mX - \check{W}\dot{x} + \frac{dm}{dt}\,(k - 1)\,\dot{x},$$

$$m\ddot{y} = mY - \check{W}\dot{y} + \frac{dm}{dt}\,(k - 1)\,\dot{y}. \qquad (5.25)$$

where $\check{W}_v$ is the value of the resistance and K is the ratio of the velocity of the changing mass and the velocity of the point is taken positive or negative according to velocities in similar or opposite direction

Set
$$W = \frac{1}{m}\left[\check{W} - (k - 1)\,\frac{dm}{dt}\right],$$

then the equation (5.25) takes the form (5.16). Therefore, in this case also, law of change of mass is again formulated.

4. Motion of a Point Having Variable Mass when it is Performed Under the Influence of Gravity and Central Forces:

The equations of motion of a heavy point having variable mass are expressed as

$$\ddot{x} = \frac{1}{m}\left[\frac{dm_1}{dt}(\alpha_1 - \dot{x}) - \frac{dm_2}{dt}(\alpha_2 - \dot{x})\right] - \tilde{W}\frac{\dot{x}}{v},$$

$$\ddot{y} = \frac{1}{m}\left[\frac{dm_1}{dt}(\beta_1 - \dot{y}) - \frac{dm_2}{dt}(\beta_2 - \dot{y})\right] - \tilde{W}\frac{\dot{y}}{v}, \qquad (5.26)$$

$$\ddot{z} = -g + \frac{1}{m}\left[\frac{dm_1}{dt}(\gamma_1 - \dot{z}) - \frac{dm_2}{dt}(\gamma_2 - \dot{z})\right] - \tilde{W}\frac{\dot{z}}{v}.$$

where $\tilde{W}$ is the medium resistance calculated per unit mass, and axis cz is directed vertically upwards.

Let only addition or ejection of particles to the exclusion of the other take place with constant average velocity u_r (both in magnitude and direction) and having projections a, b, c. We take the initial position of the point as the origin and vertical plane as the zx plane, in which u_r is confined. Then b = o and the equation (5.26) describing the motion of heavy point in vacuum can be expressed as

$$\ddot{x} = -\frac{d\beta}{dt}a, \quad \ddot{y} = 0, \quad \ddot{z} = -g - \frac{d\beta}{dt}c, \qquad (5.27)$$

where $\beta = \log \frac{m_o}{m}$

We may write five integrals of the system (5.27) as

$$\dot{x} = -a\beta + \dot{x}_o, \quad \dot{y} = \dot{y}_o, \quad \dot{z} = -c\beta - gt + \dot{z}_o,$$

$$y = \dot{y}_o t, \quad ex - az = \frac{1}{2} agt^2 + (c\dot{x}_o - a\dot{z}_o)\, t. \tag{5.28}$$

Here x_o, y_o, z_o represent the projections of the initial point of the velocity.

I.V. Meshersky[53] has discussed the above integrals and considered a few concrete examples regarding the motion of a heavy point having a variable mass.

Let us investigate how the heavy point having a variable mass would move under the action of a central force. We represent the central force by F taken with plus or minus sign; whether it is a repulsive or attractive force. Then Meshersky equation (2.5) can be written as

$$m\frac{\bar{dv}}{dt} = F\frac{\bar{r}}{r} + \frac{dm_1}{dt}(\bar{u}_1 - \bar{v}) - \frac{dm_2}{dy}(\bar{u}_2 - \bar{v}) \tag{5.29}$$

Case I: Suppose the process of change of mass takes place without impact (mass changes uniformly) i.e., $u_1 = u_2 = v_1$,

therefore, equation (5.29) becomes

$$m\frac{\bar{dv}}{dt} = F\frac{\bar{r}}{r} \tag{5.30}$$

In this case, if the force is proportional to the mass of the point, the change of the mass does not influence the motion. The problem concerning the motion of the point is resolved, if the mass of the point and the force are functions of distance of the point from the center of the force.

Case II:

Velocities of the changing masses are equalto zero

$$\bar{u}_1 = \bar{u}_2 = 0$$

Equation (5.29) is then expressed as

$$m \frac{d\bar{v}}{dt} = F \frac{\bar{r}}{r} - \frac{dm}{dt} \bar{v} \qquad (5.31)$$

Equation (5.31) admits that there are integrals and consequently the point moves in a plane through the center of the force. In this plane double sectorial velocity of the point is equal to

$$r^2\theta = \frac{c}{m} \text{ , } c = \text{const.}$$

If the mass of the point and the force depend only upon the distance between the point and the center of the force, the problem is resolved.

Case III:

Velocities of the changing masses are directed in the same direction as the velocity of the point.

$$u_1 = \nu v, \quad u_2 = \gamma v$$

Equationof the motion then becomes:

$$m \frac{dv}{dt} = F \frac{r}{r} + \left[\frac{dm_1}{dt} (\nu - 1) - \frac{dm_2}{dt} (\gamma - 1) \right] \bar{v}. \qquad (5.32)$$

Since by virtue of (5.32)

$$\frac{d}{dt} (\bar{r} \times \bar{v}) = \frac{1}{m} (\frac{dm_1}{dt} \nu - \frac{dm_2}{dt} \gamma - \frac{dm}{dt}) (\bar{r} \times \bar{v}). \qquad (5.33)$$

the follcwing integrals arise

$$\dot{yz} - z\dot{y} = c_1 f,$$

$$\dot{zx} - x\dot{z} = c_2 f,$$

$$\dot{xy} - y\dot{x} = c_3 f,$$

where

$$f = \frac{1}{m} e^{\int \frac{1}{m} (\nu dm_1 - \gamma dm_2)}$$

Therefore, the point moves in the plane

$$c_1 x + c_2 y + c_3 z = 0 \quad .$$

Double sectorial velocity is equal to

$$r^2\theta = f \sqrt{c_1^2 + c_2^2 + c_3^2}.$$

Let us assume that only ejection oraddition of the particles to the exclusion of the others takes place. In order to fix the ideas, suppose that only ejection of the particles occurs. Then

$$f = \frac{1}{m} e \int \gamma \frac{dm}{m}.$$

If γ is constant or a function of m then f can be finitely calculated. If the mass of the point γ the force can be expressed as functions of the distance, then also the theorem is provided.

In the light of the above discussion, we may investigate the motion of the comet as it approaches its periger. Assuming that the mass of the comet decreases and many be expressed as a function of distance between the comet and the sun. Then the equation of the motion is integrated if we suppose that the velocity of the inertial center of the ejected particles is either equal to the same direction in which the comet is moving. We further suppose that the ratio of these velocities is either a constant or a function of the distance between the comet and the sun.

Case IV:

The velocities of the changing masses are directed in a straight line joining the point and the center of the force.

Equation (2.5) is transformed as:

$$\frac{d}{dt}(m\bar{v}) = F\frac{\bar{r}}{r} + \left(\frac{dm_1}{dt}\gamma - \frac{dm_2}{dt}\gamma\right)\bar{r}. \qquad (5.34)$$

Consequently three area integrals will arise; the point will move in a plane passing through the center of the force; and the sectorial velocity of the point in this plane is inversely proportional to its mass.

Case V:

The projections of the reactive force, taken per unit mass, are explicit functions of time.

$$\frac{1}{m}\left[\frac{dm_1}{dt}(x_1 - \dot{x}) - \frac{dm_2}{dt}(x_2 - \dot{x})\right] = f_1(t)$$

$$\frac{1}{m}\left[\frac{dm_1}{dt}(\beta_1 - \dot{y}) - \frac{dm_2}{dt}(\beta_2 - \dot{y})\right] = f_2(t),$$

$$\frac{1}{m}\left[\frac{dm_1}{dt}(\gamma_1 - \dot{z}) - \frac{dm_2}{dt}(\gamma_2 - \dot{z})\right] = f_3(t).$$

Assuming that the central force is proportional to the mass of the point and can be expressed as a certain function of the distance

$$F = m\varphi(r),$$

then the equations of motion are obtained as

$$\ddot{x} = \varphi(r)\,\frac{x}{r} + f_1(t),$$

$$\ddot{y} = \varphi(r)\,\frac{y}{r} + f_2(t), \qquad (5.35)$$

$$\ddot{z} = \varphi(r)\,\frac{z}{r} + F_3(t).$$

Suppose the functions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ are such that $\ddot{\varphi}_i(t) = f_i(t)$ $(i=1,2,3)$, then the equations (5.35) considered with

$$x = \xi + \varphi_1(t), \quad y = \eta + \varphi_2(t), \quad z = \zeta + \varphi_3(t)$$

explains the motion of the point of constant mass in vacuum under the action of the central force whose center moves in a pre-determined manner.

Supppse only the addition or ejection of particles to the exclusion of the other takes place, the geometrical difference of the velocities of the inertial center of the changing masses and the point has a fixed direction and setting on parallel to it $\bar{u}_r = \bar{u} - \bar{v}$, then

$$f_1 = \frac{1}{m}\,\frac{dm}{dt}\,u_r, \quad f_2 = 0, \quad f_3 = 0$$

In such a case, the central force operates according to Newton's law

5. Limitations on the Dependence of the Changing Masses on the Velocity: [55]

Let us study the possible functional relationship between the changing masses m_1 and m_2 and the projections of the velocity $\dot{x}, \dot{y}, \dot{z}$. Suppose the changing masses are expressed by functions

$$m_1 = f_1 (t, x, y, z, S, \dot{x}, \dot{y}, \dot{z}),$$
$$m_2 = f_2 (t, x, y, z, S, \dot{x}, \dot{y}, \dot{z}). \qquad (5.36)$$

Where S is the path travelled by the point. Then differentiating in respect of time, we have

$$\frac{dm_1}{dt} = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x}\dot{x} + \frac{\partial f_1}{\partial y}\dot{y} + \frac{\partial f_1}{\partial z}\dot{z} + \frac{\partial f_1}{\partial S}\dot{S} + \frac{\partial f_1}{\partial \dot{x}}\ddot{x} + \frac{\partial f_1}{\partial \dot{y}}\ddot{y} + \frac{\partial f_1}{\partial \dot{z}}\ddot{z}. \qquad (5.37)$$

$$\frac{dm_2}{dt} = \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial x}\dot{x} + \frac{\partial f_2}{\partial y}\dot{y} + \frac{\partial f_2}{\partial z}\dot{z} + \frac{\partial f_2}{\partial S}\dot{S} + \frac{\partial f_2}{\partial \dot{x}}\ddot{x} + \frac{\partial f_2}{\partial \dot{y}}\ddot{y} + \frac{\partial f_2}{\partial \dot{z}}\ddot{z}.$$

$$m\ddot{x} = X + \frac{dm_1}{dt}(\alpha_1 - \dot{x}) - \frac{dm_2}{dt}(\alpha_2 - \dot{x})$$
$$m\ddot{y} = Y + \frac{dm_1}{dt}(\beta_1 - \dot{y}) - \frac{dm_2}{dt}(\beta_2 - \dot{y}), \qquad (5.38)$$
$$m\ddot{z} = Z + \frac{dm_1}{dt}(\gamma_1 - \dot{z}) - \frac{dm_2}{dt}(\gamma_2 - \dot{z}).$$

Putting in (5.37) the values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ as obtained from (5.38), we get a linear and non-homogeneous system consisting of the two equations in terms of $\frac{dm_1}{dt}$, $\frac{dm_2}{dt}$ the determinant of this system is

$$m - \frac{\partial f_1}{\partial x}(\alpha_1 - \dot{x}) - \frac{\partial f_1}{\partial y}(\beta_1 - \dot{y}) - \frac{\partial f_1}{\partial z}(\gamma_1 - \dot{z})$$

$$\frac{\partial f_1}{\partial \dot{x}}(\alpha_2 - \dot{x}) + \frac{\partial f_1}{\partial \dot{y}}(\beta_2 - \dot{y}) + \frac{\partial f_1}{\partial \dot{z}}(\gamma_2 - \dot{z})$$

$$- \frac{\partial f_2}{\partial x}(\alpha_1 - \dot{x}) - \frac{\partial f_2}{\partial y}(\beta_1 - \dot{y}) - \frac{\partial f_2}{\partial z}(\gamma_1 - \dot{z})$$

$$m + \frac{\partial f_2}{\partial x}(\alpha_2 - \dot{x}) + \frac{\partial f_2}{\partial y}(\beta_2 - \dot{y}) + \frac{\partial f_2}{\partial z}(\gamma_2 - \dot{z})$$

Substituting in (5.38) values of $\frac{dm_1}{dt}$ and $\frac{dm_2}{dt}$ obtained from (5.37) we get a system of three linear equations in terms of $\ddot{x}, \ddot{y}, \ddot{z}$ whose determinant Δ_2 is

$$m - \frac{\partial f_1}{\partial x}(\alpha_1 - \dot{x}) + \frac{\partial f_2}{\partial x}(\alpha_2 - \dot{x}) - \frac{\partial f_1}{\partial y}(\alpha_1 - \dot{x}) + \frac{\partial f_2}{\partial y}(\alpha_2 - \dot{x})$$

$$- \frac{\partial f_1}{\partial z}(\alpha_1 - \dot{x}) + \frac{\partial f_2}{\partial z}(\alpha_2 - \dot{x})$$

$$- \frac{\partial f_1}{\partial x}(\beta_1 - \dot{y}) + \frac{\partial f_2}{\partial x}(\beta_2 - \dot{y}) \quad m - \frac{\partial f_1}{\partial y}(\beta_1 - \dot{y}) + \frac{\partial f_2}{\partial y}$$

$$(\beta_2 - \dot{y}) - \frac{\partial f_1}{\partial z}(\beta_1 - \dot{y}) + \frac{\partial f_2}{\partial z}(\beta_2 - \dot{y})$$

$$- \frac{\partial f_1}{\partial x}(\gamma_1 - \dot{z}) + \frac{\partial f_2}{\partial x}(\gamma_2 - \dot{z}) \quad - \frac{\partial f_1}{\partial y}(\gamma_2 - \dot{z}) + \frac{\partial f_2}{\partial y}(\gamma_2 - \dot{z})$$

$$m - \frac{\partial f_1}{\partial z}(\gamma_1 - \dot{z}) + \frac{\partial f_2}{\partial z}(\gamma_2 - \dot{z})$$

Δ_2 is a polynomise of the third degree in an

$$\Delta_2(m) = m^3 + Am^2 + Bm + C. \qquad (5.39)$$

Set $C = \Delta_2(0)$, therefore, coefficient C is equal to the determinant

$$-\frac{\partial f_1}{\partial \dot{x}}(x_1 - \dot{x}) + \frac{\partial f_2}{\partial \dot{x}}(x_2 - \dot{x}) \quad -\frac{\partial f_1}{\partial \dot{y}}(x_1 - \dot{x}) + \frac{\partial f_2}{\partial \dot{y}}(x_2 - \dot{x})$$

$$-\frac{\partial f_1}{\partial \dot{z}}(x_1 - \dot{x}) + \frac{\partial f_2}{\partial \dot{z}}(x_2 - \dot{x})$$

$$-\frac{\partial f_1}{\partial \dot{x}}(\beta_1 - y) + \frac{\partial f_2}{\partial \dot{x}}(\beta_2 - \dot{y}) \quad -\frac{\partial f_1}{\partial \dot{y}}(\beta_1 - \dot{y}) + \frac{\partial f_2}{\partial \dot{y}}(\beta_2 - \dot{y}) -$$

$$-\frac{\partial f_1}{\partial \dot{z}}(\beta_1 - y) + \frac{\partial f_2}{\partial \dot{z}}(\beta_2 - \dot{y})$$

$$-\frac{\partial f_1}{\partial \dot{x}}(\gamma_1 - \dot{z}) + \frac{\partial f_2}{\partial \dot{x}}(\gamma_2 - \dot{z}) \quad -\frac{\partial f_1}{\partial \dot{y}}(\gamma_1 - \dot{z}) + \frac{\partial f_2}{\partial \dot{y}}(\gamma_2 - \dot{z})$$

$$-\frac{\partial f_1}{\partial \dot{z}}(\gamma_1 - \dot{z}) + \frac{\partial f_2}{\partial \dot{z}}(\gamma_2 - \dot{z})$$

It is easily seen that $C = 0$

Moreover,

$$B = \frac{d\Delta_2(m)}{dm}\Big|_{m=0}$$

Consequently, coefficient B is equal to the sum of the three diagonal determinants of the second order, obtained from the matrix of the determinant $\Delta_2(0)$. It may be written in the ferm

$$B=\begin{vmatrix} \alpha_1 - \dot{x}\,\dfrac{D(f_1, f_2)}{D(y, z)} & \alpha_2 - \dot{x} \\ \beta_1 - \dot{y}\,\dfrac{D(f_1, f_2)}{D(z, x)} & \beta_2 - \dot{y} \\ \gamma_1 - \dot{z}\,\dfrac{D(f_1, f_2)}{D(x, y)} & \gamma_2 - \dot{z} \end{vmatrix} \tag{5.40}$$

elements in the second column are functional determinants, computed from the functions f_1 and f_1 in respect of the indicated projections of the velocity.

We now determine A and get

$$A = \frac{1}{2}\left.\frac{d^2\Delta_2(m)}{dm^2}\right|_{m=0}$$

$$A = \frac{\partial f_1}{\partial x}(-\alpha_1 - \dot{x}) - \frac{\partial f_1}{\partial y}(\beta_1 - \dot{y}) - \frac{\partial f_1}{\partial z}(\gamma_1 - \dot{z}) +$$

$$+ \frac{\partial f_2}{\partial x}(\alpha_2 - \dot{x}) + \frac{\partial f_2}{\partial y}(\beta_2 - \dot{y}) + \frac{\partial f_2}{\partial z}(\gamma_2 - \dot{z}) \tag{5.41}$$

By similar considerations, we will find that

$$\Delta_1 = m^2 + Am + B \tag{5.42}$$

therefore

$$\Delta_2 = m\Delta_1 \tag{5.43}$$

If we exclude the special case from consideration when the linear equations in $\frac{dm_1}{dt}$ and $\frac{dm_2}{dt}$ or the equations in terms of $\dot{x},\dot{y},\dot{z}$ are homogeneous, then the functions (5.36) determine the change of the mass only if $\Delta_1 \neq 0$.

Suppose the velocities of the changing masses are equal to zero, then

$$\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = 0, \gamma_1 = \gamma_2 = 0;$$

Hence B = 0 and

$$\Delta_1 = m \left[m + \frac{\partial(f_1 - f_2)}{\partial \dot{x}} \dot{x} + \frac{\partial(f_1 - f_2)}{\partial \dot{y}} \dot{y} + \frac{\partial(f_1 - f_2)}{\partial \dot{z}} \dot{z} \right].$$

Since $m = m_0 + i_1 - f_2$, the above result, treating Δ_1 to be zero, is equivalent to

$$\frac{\partial m}{\partial \dot{x}} \dot{x} + \frac{\partial m}{\partial \dot{y}} \dot{y} + \frac{\partial m}{\partial \dot{z}} \dot{z} = - m. \qquad (5.44)$$

(5.44) shows that m is a homogeneous function of the velocities (negative) and it is of the first order. Therefore, if the Δ_1 expression for the mass m identically satisfies the equation

$$f(m\dot{x}, m\dot{y}, m\dot{z}, x, y, z, s, t) = 0$$

then f is equal to zero. In case of the rectilinear motion we hzve

$$f(m\dot{x}, x, t) = 0$$

$$m = \frac{1}{\dot{x}} f(x, t),$$

where $\check{f}(x,t)$ - which is a function of x and t.

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CHAPTER-II

1. LAW OF CHANGE OF THE PRINCIPAL VECTOR OF MOMENTUM OF A SYSTEM OF POINTS HAVING A VARIABLE MASS.

1. Change of the Principal Vector of Momentum of a System having a Variable Mass:

We will consider a system of points having variable masses, which consists of finite or infinite number of points. Equation (1.2) in Chapter-I characterizes the motion of each material point.

Therefore, the motion of j the point having a variable mass is

$$m_j \frac{\bar{dv}_j}{dt} = \bar{F}'_j + \bar{F}''_j + \bar{R}_j, \qquad (1.1).....$$

where $\bar{F}'_j$ denotes the component of all external forces acting on j, the point. $\bar{F}''_j$ - component of all internal forces exerted on the point from the direction of other points of the investigated mechanical system, $\bar{R}_j$ - reactive force exerted on j, the point.

By (1.3) of Chapter I the reactive force may be expressed as

$$\bar{R}_j = -\sum_{k} m_{kj} \bar{\omega}^r_{kj} + J_j + \Phi_j, \qquad (1.2).....$$

where the index k signifies all the particles moving in the material point, and $\bar{\omega}^{r}_{kj}$ denotes the average accelerations of these particles.

The coriliso force is found from the equation (1.4) in Chapter-I.

$$\bar{J}_j = -2 \sum_k \bar{\omega}_j \times m_{kj} \, u^r_{kj} \qquad (1.3)$$

The summation in (1.3) is taken over all the particles moving in j, the point, W_j - is the angular velocity of the auxiliary system of the coordinates whose origin is in j, the point, u^r_{kj} - the average velocities of the particles.

Impulsive force for j, the material point by (1.7) in Chapter-I is

$$\bar{\Phi}_j = - \sum_j \frac{d\mu_{ij}}{dt} \Delta \bar{u}^r_{ij} \qquad (1.4)$$

where the summation is taken over all the directions of the impulsive change in the velocity of particles in j, the point; $\frac{d\mu_{ij}}{dt}$ - rate of flow of particle in i-directions; $\Delta \bar{u}^r_{ij}$ - impulsive change in the average velocity of the particles. The impulsive force can be expressed as a surface integral over the solid angle round j, the point, as shown in Chapter-I.

The vector (quantity)

$$m_j \bar{v}_j = \bar{Q}_j \tag{1.5}$$

is called the momentum of j, the point having a variable mass. From the equation (1.1) we obtain the law of change of momentum of point in the form of

$$\frac{d^*}{dt} \bar{Q}_j = \bar{F}_j + \bar{F}_j + \bar{R}_j. \tag{1.6}$$

$$\frac{d^*}{dt} \Psi \left[m_j, t, r_j(t), v_j(t) \right]$$

is partial derivative in respect of time of the function Ψ in which m_j and t are independent variables. From the instant of time when the mass cases to change, $\frac{d^*}{dt}$ is converted into the total derivative in respect of time.

A.I. Puree (45) was the first to use the derivative in respect of time market with superscript star for the consolidated masses. Equation (1.6) may be rewritten as :

$$d^*Q_j = (\bar{F}_j' + \bar{F}_j'' + \bar{R}_j)\, dt. \tag{1.7}$$

where d^*Q_j - is partial differential of the momentum of j, the point computed for the consolidated, masses. A point having a variable mass is said to be consolidated at time t, if the mass ceases to change from this instant, although the terminology of "point" suggested by F.R. Gantuseder and L.M. Jevin (11,12) for the rocket is a recognised one. Therefore, the law of change of momentum of point having a variable mass may be reformulated in the following way.

Elementary increment in the momentum of the consolidated point having a variable mass is equal to the geometrical sum of the moments of all forces acting on the point.

Writing equation (1.6) for all points of the system and adding, we obtain:

$$\frac{d^*}{dt}\bar{Q} = \bar{F} + \bar{R}. \qquad (1.8)$$

where $\bar{F} = \sum_j \bar{F}_j$ - principal vector of all external forces acting on the system; $\bar{R} = \sum_j \bar{R}_j$ - principal vector of reactive forces.

We have

$$\bar{R} = -\sum_{k,j} m_{kj}\bar{w}^r_{kj} + \bar{J} + \bar{\Phi} \qquad (1.9)$$

The summationin the first member onright side of (1.9) is takenover all the particles moving in investigated points; and

$$\bar{J} = -2 \sum_{k,j} \bar{\omega}_j \times m_{kj} \bar{u}^r_{kj} \qquad (1.10)$$

— = principal vector of the Coriliso forces,

Finally,

$$\bar{\Phi} = \sum_j \bar{\Phi}_j = - \sum_j \sum_j \frac{d\gamma_{ij}}{dt} \Delta \bar{u}^r_{ij} \qquad (1.11)$$

- principal vector of all the impulsive forces. In this the summation is taken over all the directions of the impulsive change in the velocity of particles (index i) in the entire investigated mechanical system (index j).

In case of the system of points there is a continuous medium then the principal vector of the impulsive forces $\bar{\Phi}$ may be expressed as the fifth order of the integral

$$\bar{\Phi} = \iiint_{(\tau)} \iint_{(\sigma)} \bar{\varphi}(x, y, z; \theta, \psi)\, d\sigma\, d\tau, \qquad (1.12)$$

$$d\sigma = \sin\theta\, d\theta\, d\psi, \quad d\tau = dxdydz,$$

where $\iint_{(\sigma)}$ is integrated over the solid angle round the geometrical point of the medium with coordination x,y,z; integral over the medium: Function $(x,y,z; \iiint_{(\tau)})$ is specific value of $\bar{\varphi}(x, y, z, \theta, \psi)$

$$- \frac{d\gamma_{ij}}{dt} \Delta \bar{u}^r_{ij}$$

Equation (1.8) states the law of change of the principal vector of momentum for the system of points having a variable mass. The law can be written as

$$d^*\bar{Q} = (\bar{F} + \bar{R})\, dt \qquad (1.13)$$

If the process of change of mass completely stops at time t, the system is then said to be consolidated at this time. The elementary increment in the principal vector of the momentum of the consolidated system of the points is equal to the sum of moments of the external forces and the forces of reaction.

2. Equation of Motion of the Inertial center of a System of Points Having Variable Masses:

We will investigate how the center of the inertial of the system moves. The radius vector of the inertial center is given by

$$mr_c = \sum_j m_j r_j, \qquad (1.14)$$

where

$$m = \sum_j m_j$$

is the total mass of the system of points; r_j- radium vector of j, point in respect of the origin, taken as r_c,

Taking a fixed point as the origin, find $\frac{d^*}{dt}$ from equation (1.14)

$$m \frac{d^*}{dt} r_c - Q \tag{1.15}$$

Differentiating once more and using(1.18), we obtain

$$m \frac{d^{2*}}{dt^2} \bar{r}_c = \bar{F} + \bar{R} \tag{1.16}$$

which provides the vector equation of the inertial center of the system of points having a variable mass.

Now we will determine the relationship between $\frac{d^{2*}}{dt^2} \bar{r}_c$ and the acceleration of the inertial center, $\bar{w}_c$. The radius vector of the inertial center in respect of a fixed origin is given by

$$\bar{r}_c = \frac{1}{m} \sum_j m_j \bar{r}_j. \tag{1/17}$$

Differentiating it in respect of time, we obtain

$$\bar{\dot{r}}_c = \frac{1}{m} \sum_j m_j \dot{\bar{r}}_j + \frac{1}{m} \sum_j \dot{m}_j \bar{r}_j - \frac{\dot{m}}{m^2} \sum_j m_j \bar{r}_j.$$

Since $\bar{r}_c = \bar{v}_c$, $\bar{v}_c$ - being the velocity of the inertial center and

$$\frac{1}{m} \sum_j n_j \dot{\bar{r}}_j = \frac{d^*}{dt} \bar{r}_c$$

$$-\frac{\dot{m}}{m^2}\sum_j m_j\bar{r}_j = -\frac{\dot{m}}{m^2}\cdot n\bar{r}_c = -\frac{1}{m}\sum_j \dot{m}_j\bar{r}_c.$$

then

$$\bar{v}_c = \frac{d^*}{dt}\bar{r}_c + \frac{1}{m}\sum_j \dot{m}_j\,(\bar{r}_j - \bar{r}_c). \qquad (1.18)$$

The quantity $\frac{d^*}{dt}\bar{r}_c$ represents the velocity of the inertial center of the consolidated system of points having a variable mass. Therefore, equation (1.18) provides the relationship between the the actual velocity of the inertial center of the system of point and the velocity of the inertial center of the consolidated system.

Differentiating (1.18) in respect of time, we obtain

$$\bar{w}_c = \dot{\bar{v}}_c = \frac{1}{m}\sum_j m_j\ddot{\bar{r}}_j + \frac{1}{m}\sum_j \dot{m}_j\dot{\bar{r}}_j - \frac{\dot{m}}{m^2}\sum_j m_j +$$

$$+\frac{1}{m}\sum_j \ddot{m}_j\,(\bar{r}_j - \bar{r}_c) + \frac{1}{m}\sum_j \dot{m}_j\,(\dot{\bar{r}}_j - \dot{\bar{r}}_c) -$$

$$-\frac{\dot{m}}{m^2}\sum_j \dot{m}_j\,(\bar{r}_j - \bar{r}_c)$$

As the following equality holds

$$-\frac{\dot{m}}{m^2}\sum_j m_j\dot{\bar{r}}_j - \frac{\dot{m}}{m^2}\sum_j \dot{m}_j(\bar{r}_j - \bar{r}_c) = -\frac{\dot{m}}{m^2}\frac{d}{dt}(m\bar{r}_c) + \left(\frac{\dot{m}}{m}\right)^2\bar{r}_c = -\frac{\dot{m}}{m}\dot{\bar{r}}_c,$$

therefore,

$$w_c = \frac{d^{2*}}{dt^2}\bar{r}_c + \frac{1}{m}\sum_j \dot{m}_j (\bar{r}_j - \bar{r}_c) + \frac{2}{m}\sum_j m_j (\bar{v}_j - \bar{v}_c). \quad (1.19)$$

which establishes the relationship between w_c and

In case of the continuous medium the results (1.18) and (1.19) will be expressed as

$$v_c = \frac{d^*}{dt} r_c + \frac{1}{m}\iiint_{(\tau)} (\bar{r} - \bar{r}_c)\frac{\partial \rho}{\partial t}\, d\tau, \quad (1.20)$$

and

$$w_c = \frac{d^{2*}}{dt^2} r_c + \frac{1}{m}\iiint_{(\tau)} \left[(\bar{r} - \bar{r}_c)\frac{\partial^2 \rho}{\partial t^2} + 2(v - v_c)\frac{\partial \rho}{\partial t}\right] d\tau \quad (1.21)$$

Where $\rho(x,y,z,t)$ - is the density function of the medium; the triple integrals will be taken over the medium. In case of the surface of the linear medium the results will involve double or single integrals.

From equation (1.19) we see that generally

$$w_c \neq \frac{d^{2*}}{dt^2} r_c.$$

Therefore, equation (1.16) differs from the equation (1.2) (of Chapter-I) of motion of point having a variable mass. Thus, in mechanics of the variable masses, inertial center of the system of points does not move in the same way as the material point whose mass is equal to the mass of all the material points entering into the

investigated system and on which a force equal to the geometrical sum of all forces acting on the points of the system is exerted.

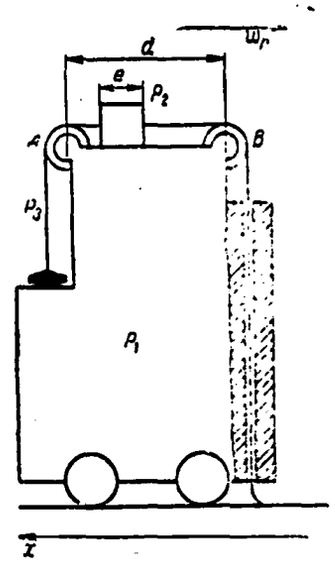

Fig.4

Рис. 4

3. Example:

We will consider an example showing application of the inertial center motion.

A trolley of weight P_1 (Fig.4) moves on a rough plane. A load weighing P_2 has a relative (average) acceleration W^r. The load is, on both Fig.4 sides, fastened with a heavy flexible chain whose weight per unit length is γ. On the left, the chain passes over a block A and rests on the trolley. On the right, the chain goes over a block B then passing through a smooth vertical tube rest onthe ground. Neglecting the rolling function of the mass of the blocks and the friction in axel (shafts), and supposing the AB to be ideal, find the acceleration of the

trolley if the weight of claim on the trolley at a given moment of time is equal to P_3. The distance AB is d and the width of load is e.

In the present case, the external forces and the reactive forces will be acting vertically and, therefore, projecting inertial center equation (1.16) on x, axis we obtain

$$\frac{d^{2*}}{dt^2} x_c = 0.$$

therefore, it follows that

$$\sum m\ddot{x} = 0$$

or more precisely

$$\left[P_1 + P_3 - \gamma(d - e)\right] \ddot{x} + \left[P_2 + \gamma(d - e)\right] (\ddot{x} - w^r) = 0$$

therefore, the acceleration of the trolley is

$$\ddot{x} = \frac{P_2 + \gamma(d - e)}{P_1 + P_2 - P_3} w^r$$

2. LAW OF CHANGE OF KINETIC MOMENT OF A SYSTEM OF POINTS HAVING VARIABLE MASSES:

1. Change of the Kinetic moment of a System of Point Having Variable Masses:

Multiply vectorially equation (1.1) by the radius vector of j, th point with respect to a fixed origin $\bar{r}_j$ and adding such equation for all material points of the system, we have

$$\sum_j m_j \left(\bar{r}_j \times \frac{d\bar{v}_j}{dt} \right) = \bar{L} + \bar{M}. \qquad (2.1)$$

where

$$\bar{L} = \sum_j \bar{r}_j \times \bar{F}_j$$

- principal moment of the external forces acting on points of the system

$$\bar{M} = -\sum \bar{r} \times m\bar{w}_r + \bar{H} + \bar{K} \qquad (2.2)$$

- principal moment of reactive forces; in which

$$\bar{H} = \sum_j \bar{r}_j, \times \bar{J}_j, \quad \bar{K} = \sum_j \bar{r}_j \times \Phi_j.$$

$\bar{H}$ = principal moment of Coriliso forces. principal moment of $\bar{K}$-impulsive forces

If the system of points is a continuous medium, the $\bar{L}$, $\bar{H}$

and $\sum \bar{r} \times m\bar{w}_r$ may be expressed as volume integrals. The principal moment of impulsive forces, in this case, would be a fifth order integral

$$K = \iiint_{(\tau)} \iint_{(\sigma)} \tilde{r} \times \tilde{\varphi}(x, y, z; \theta, \psi)\, d\sigma\, d\tau , \qquad (2.3)$$

$$d\sigma = \sin\theta\, d\theta\, d\psi, \quad d\tau = dxdydz.$$

$\iint_{(\sigma)}$ is surface integral over the solid angle found the geometrical point of the medium with coordinates (x,y,z); $\iiint_{(\tau)}$ is volume integral considering properties of the partial derivatives and using (2.1), we obtain

$$\frac{d^*}{dt} \bar{l} = \bar{L} + \bar{M} \qquad (2.4)$$

where

$$\bar{l} = \sum_j \bar{r}_j \times m_j \bar{v}_j \qquad (2.5)$$

- Kinetic moment of the system of points having a variable mass, calculated in respect of a fixed point.

(2.4) states the law of change of the Kinetic moment. The derivative with respect to time of the Kinetic moment with respect to a fixed point of consolidated system is equal to the sum of the principal moment, in respect of a fixed point of the external and the reaction forces.

We will now develop the law of change of the kinetic

moment of the system of points having a variable mass in respect of the center of mass of the system. We have

$$\bar{l} = \bar{l}_c + \sum_j \bar{r}_c \times m_j \bar{v}_j, \quad \bar{l}_c = \sum_j (\bar{r}_j - \bar{r}_c) \times m_j \bar{v}_j,$$

where $\bar{l}_c$ is the kinetic moment of the system in respect of the center of inertia

$$\frac{d^*}{dt} \sum_j \bar{r}_c \times m_j \bar{v}_j = \bar{r}_c \times \frac{d^*}{dt} \bar{Q}.$$

By virtue of (1.8) and (2.4), we have

$$\frac{d^*}{dt} \bar{l}_G = \bar{L}_c + \bar{M}_c. \tag{2.6}$$

where

$$\bar{L}_c = \sum_j (\bar{r}_j - \bar{r}_c) \times \bar{F}_j, \quad \bar{M}_c = \sum_j (\bar{r}_j - \bar{r}_c) \times \bar{R}_j$$

are the principal moments of the external and reaction force in respect of the inertial center of the system of points having a variable mass.

therefore

$$l_c = \sum_j (\bar{r}_j - \bar{r}_c) \times m_j (\bar{v}_j - \bar{v}_c), \tag{2.7}$$

$$\tilde{l}_c = \sum_j (\bar{r}_j - \bar{r}_c) \times m_j (\bar{v}_j - \frac{d^*}{dt} \bar{r}_c). \qquad (2.8)$$

2. Example Showing a Roating Winch with Unwounding and Wounding Chains.

We will consider a practical application which uses equation (2.6). An absolutely flexible chain, whose weight per unit length is γ_1, is unwound from the barrel of the radius r_1. On the shaft, whose radius is r_2, are wound two absolutely flexible-chains which rest on the ground. The weight per unit length of each chain is γ_2. The shaft has a constant torque L_1.

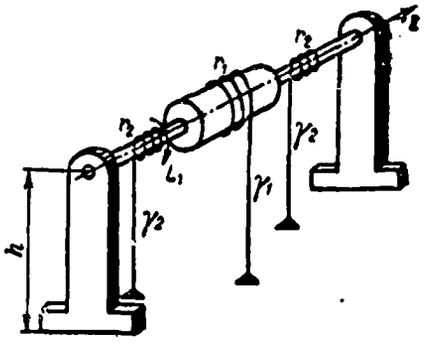

Рис. 5

Fig.5

Neglecting the bearing friction and transverse dimensions of chain, we determine the angular velocity of the rotation of the shaft, if its moment of inertia in respect of rotation axis is and initially the shaft is at rest. Height of rotation axis

is l_2 and initially the shaft is at rest. Height of rotation axis from the ground is h. Initially the chain of length l_1 is wound on the barrel, and the chains, each of l_2 length, are wound on the shaft on both the sides of the barrel.

Projecting equation (2.6) on z -axis, we obtain

$$\frac{d^*}{dt} I_z = L_z + M_z. \tag{2.9}$$

Where is the angle of rotationof the shaft

$$\left[I_z + \frac{\gamma_1}{g} r_1^2 (h + l_1 - r_1 \varphi) + L_2 + L_3 \right.$$

The projection of the principal moment of external forces on the axis of Z gives

$$L_2 = L_1 + h (\gamma_1 r_1 - 2 \gamma_2 r_2) + L_2 + L_3$$

Where L_2 - moment of the weight of chain wound on the barrel in respect of z- axis; L_3 - moment of the weight of chain wound on the shaft in respect of z-axis. Assuming that the chain is very thin , we have

$$L_2 = \int_0^{\frac{l_1}{r_1} - } \gamma_1 r_1^2 \cos \alpha \, d\alpha = \gamma_1 r_1^2 \sin \left(\frac{l_1}{r_1} \varphi \right)$$

and

$$L_3 = -2 \int_0^{\frac{l_2}{r_2}+\varphi} \gamma_2 r_2^2 \cos\alpha \, d\alpha = -2\gamma_2 r_2^2 \sin\left(\frac{l_2}{r_2}+\varphi\right)$$

Finally

$$L_z = L_1 + h(\gamma_1 r_1 - 2\gamma_2 r_2) + \gamma_1 r_1^2 \sin\left(\frac{l_1}{r_1} - \varphi\right) -$$

$$- 2\gamma_2 r_2^2 \sin\left(\frac{l_2}{r_2} + \varphi\right).$$

In the present example, the reaction forces consist of the impulsive forces so that $M_z = K_z$. The impulsive forces exerted on are numerically equal to

$$\frac{\gamma_2}{g} r_2^2 \varphi^2$$

The impulsive force exerted on the strong part of the chain vanishes. Since the chain is flexible, the reaction of the ground on the chain link which rests on it, is not transmitted to the entire chain as if the chain links are cut off from the rest of the chain and simply got off with zero velocity.

Therefore, it follows that

$$K_z = -2\frac{\gamma_2}{g} r_2^3 \varphi^2.$$

Equation (2.9) is then expressed as

$$\left[I_z + \frac{\gamma_1}{g} r_1^2 (h+l_1-r_1\varphi) + 2 \frac{\gamma_2}{g} r_2^2 (h+l_2+ r_2\varphi) \right] \ddot{\varphi} =$$

$$= L_1 + h(\gamma_1 r_1 - 2 \gamma_2 r_2) + \gamma_1 r_1^2 \sin (\frac{l_1}{r_1} - \varphi) -$$

$$- 2\gamma_2 r_2^2 \sin (\frac{l_2}{r_2}) - 2 \frac{\gamma_2}{g} r_2^3 \dot{\varphi}^2 \qquad (2.10)$$

Since

$$\ddot{\varphi} = \frac{1}{2} \frac{d\dot{\varphi}^2}{d\varphi}$$

then (2.10) becomes

$$\frac{d\dot{\varphi}^2}{d\varphi} + \frac{4 \frac{\gamma_2}{g} r_2^3}{I_z + \frac{\gamma_1}{g} r_1^2(h+l_1) + 2 \frac{\gamma_2}{2} r_2^2 (h+l_2) - \frac{\gamma_1 r_1^3 - 2\gamma_2 r_2^3}{g} \varphi} \dot{\varphi}^2 = f(\varphi)$$

...(2.11)

where

$$f(\varphi) = 2 \frac{I_1 + h(\gamma_1 r_1 - 2 \gamma_2 r_2) + \gamma_1 r_1^2 \sin (\frac{l_1}{r_1} - \varphi) - 2 \gamma_2 r_2^2 \sin (\frac{l_2}{r_2} + \varphi)}{I_z + \frac{\gamma_1}{g} r_1^2 (h+l_1) + 2 \frac{\gamma_2}{g} r_2^2 (h+l_2) - \frac{\gamma_1 r_1^3 - 2\gamma_2 r_2^3}{g} \varphi}$$

Equation (2.11) is the linear of the first often in φ^2 for argument φ, therefore

$$\varphi^2 = (a - b\varphi)^{\varepsilon} \int_0^{\varphi} f(\varphi)\,(a - b\varphi)^{-\varepsilon}\,d\varphi,$$

where

$$a = I_z + \frac{\gamma_1}{\varepsilon} r_1^2 (h + l_1) + \frac{\gamma_2}{\varepsilon} r_2^2 (h + l_2),$$

$$b = \frac{\gamma_1 r_1^3 - 2\gamma_2 r_2^3}{\varepsilon}, \qquad = \frac{4\gamma_2 r_2^3}{\gamma_1 r_1^3 - 2\gamma_2 r_2^3}$$

3. PROBLEM OF THE TWO MATERIAL POINTS HAVING VARIABLE MASSES:

The present discussion deals with the two celestial bodies having variable masses, the process of whose change of mass can be reduced to Meshersky's case (whose mass changes according to Meshersky's law). In other words, we will deal with two points having variable masses so that the particles until ejection remain in the material points, and similarly, the joining particles after addition have velocities equal to the velocity of the material point to which they join.

1. Meshersky's two laws of Change of Mass:

Suppose the relative velocities of the inertial centers of changing masses in relation to the corresponding material points having variable masses are zero. Denote the masses of the points by m_1 and m_2, and their radius vectors in respect of certain stationary point

by r_1 (x_1, y_1, z_1) and r_2 (x_2, y_2, z_2) then equation (2.5) of Chapter I yields:

$$m_1 \frac{d^2\bar{r}_1}{dt^2} = - \frac{k^2 m_1 m_2 (\bar{r}_1 - \bar{r}_2)}{|\bar{r}_1 - \bar{r}_2|^3} \qquad (3.1)$$

and

$$m_2 \frac{d^2\bar{r}_2}{dt^2} = - \frac{k^2 m_1 m_2 (r_2 - r_1)}{|\bar{r}_1 - \bar{r}_2|^3} \qquad (3.2)$$

where k^2 is constant of the attraction.

It has an application in the problem of the two celestial bodies whose masses decrease on account of radiation (double stars: for example, the system of the sun and the comet so that the mass lost by the comet leaves it with infinitesimally small relative velocity).

Cancel m_1 and m_2 from equation (3.1) and (3.2) respectively. Substract (3.1) from (3.2). Let the radius vector of a point having mass m_1 in respect of material point having mass m_2 be denoted by $\bar{r}$ (x,y,z).

$$\bar{r} = \bar{r}_2 - \bar{r}_1; \; x = x_2 - x_1, \; y = y_2 - y_1, \; z = z_2 - z_1;$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (3.3)$$

and transposing all terms to the left, we obtain

$$\frac{d^2\bar{r}}{dt^2} + \frac{k^2 (m_1 + m_2)\, \bar{r}}{r^3} = 0 \qquad (3.4)$$

In the present case, the differential equation of the relative motion (2.4) is identical with equation obtained for the constant mass.

Since equation (3.4) assumes the vector integral of the areas, the point will move in a plane and, therefore, we may suppose that z=0. Equation (3.4) on projection becomes:

$$\ddot{x} + \frac{\mu^2 x}{(x^2 + y^2)^{\frac{3}{2}}} = 0, \quad y + \frac{\mu^2 y}{(x^2 + y^2)^{\frac{3}{2}}} = 0 \qquad (3.5)$$

where $\mu^2 = k^2(m_1 + m_2)$.

Let

$$\mu^2 = \frac{1}{at + b} \qquad (3.6)$$

where a and b are constants; t - time. This is known as First Law of Meshersky, and was worked out by Meshersky in 1902 (49). Making the transformations

$$\xi = \frac{x}{at + b}, \eta = \frac{y}{at + b}, \tau = \frac{1}{a(at + b)}, \qquad (3.7)$$

We obtain the differential equations connecting ξ, η and τ

$$\frac{d^2\xi}{d\tau^2} + \frac{\xi}{(\xi^2+\eta^2)^{\frac{3}{2}}} = 0, \quad \frac{d^2\eta}{d\tau^2} + \frac{\eta}{(\xi^2+\eta^2)^{\frac{3}{2}}} = 0.$$

Equation (3.8) shows that the investigated problem of the two points having variable masses reduces to the classical problems of the two material point with constant masses and held (pulled) together according to Newton's law.

We now develop Meshersky's second law of change of function μ^2 (54)

$$\mu^2 = -\frac{1}{\sqrt{at^2+bt+c}}, \qquad (3.9)$$

where a and b are

It is easy to see that formula (3.6) is a special case of (3.9)

Using the tronsformations:

$$\xi = \mu^2 x, \quad \eta = \mu^2 y, \quad d\tau = \mu^4\, dt \qquad (3.10)$$

we obtain ξ and η

$$\frac{d^2\xi}{d\tau^2} = -\frac{\xi}{(\xi^2+\eta^2)^{\frac{3}{2}}} - n\xi, \quad \frac{d^2\eta}{d\tau^2} = -\frac{\eta}{(\xi^2+\eta^2)^{\frac{3}{2}}} = n\eta$$

.....(3.11)

where $m = ac - \frac{b^2}{4}$

Consequently, the problem of motion of the two points having variable masses and mutually attracting (pulling) each other according to Newton's law reduces to the motion of one point with constant of the two forces namely, Newton's force of attraction and the other is proportional to the distance. Equation (3.11) is integrated Coordinates are expressible as elliptic functions of time.

2. Motionof Changing Masses when the Average (Relative) Velocities are Zero.

Transform equations (3.5) by introducing a new variable τ,

$$d\tau = \mu dt.$$

This transforms the equations to

$$\frac{d^2x}{d\tau^2} + \frac{1}{\mu}\frac{d\mu}{d\tau}\frac{dx}{d\tau} + \frac{x}{(x^2+y^2)^{\frac{3}{2}}} = 0$$

$$\frac{d^2y}{d\tau^2} + \frac{1}{\mu}\frac{d\mu}{d\tau}\frac{dy}{d\tau} + \frac{y}{(x^2+y^2)^{\frac{3}{2}}} = 0$$

We observe that the monotonic property of μ taken as a function of the variable μ follows from the monotonic property of τ as a function of time.

We obtained the following result of A.G. Lapina (43). In case of the growing masses the problem of the two points whose motion

is determined by equations (3.5) reduces to the motion of a point with constant masses under the action of the force of attraction according to Newton's law and the resistance which is expressed as a product of velocity and a certain function of time. In case of decreasing masses the last (force) is replaced by accelerating the tangential force expressed as a function (of the same type).

The work carried out by G.N. Dubshin on the system (3.5) was published in "The Journal of Astronomy " during the years 1925-1930 [15-19]. We will consider only one interesting result published in his first report.

We will assume that μ is a function of time only and hence it is positive single valued, continuous and differentiable continuity of μ (t) ensures in any closed interval of time.

System (3.5) admits an area integral which in polar coordinates has the form

$$r^2 \dot{\theta} = c, \quad c = \text{const.} \tag{3.12}$$

where θ is polar angle. Let $C \neq 0$.

Projecting the vector equation (3.4) on the radius vector, we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu^2(t)}{r^2} \tag{3.13}$$

by (3.12), we get

$$\ddot{r} = \frac{c^2}{r^3} - \frac{\mu^2(t)}{r^2} \tag{3.14}$$

Since $\mu(t)$ is a bound function, in view of equation (3.14) we see that r will acquire infinitely large values only when t tends to infinity. The function r (t) is bound in any finite interval of time (C to D). It is singly valued because $\mu(t)$ is singly valued and differentiable.

We will develop Benz formula for the central force

Set $$u = \frac{1}{r} \qquad (3.15)$$

by virtue of (3.12) and (3.15) we have

$$dt = \frac{1}{cu^2}\, d\theta \qquad (3.16)$$

The symbols with prime denote derivatives with respect to θ. Multiply equation (3.14) by r^2 and change it into new variables. It gives

$$u'' + u = \frac{1}{c^2} F(\theta) \qquad (3.17)$$

Where $F(\theta) = \mu^2 [t(\theta)]$.

Since r (t) is a single valued function therefore, by virtue of (3.12) θ is also a single valued function of time. Consequently $\theta(t)$ is a single valued function. It follows that u willl be single valued function of a polar angle θ.

Suppose β is a root of the equation $u(\theta) = 0$. The value $u = 0$ makes r infinitely large and consequently t is infinitely large. Since u is a single valued function of the angle θ root β cannot be of

a fractional multiplicity. Therefore u (β) and higher order derivatives for θ (at $\theta = \beta$) will be bounded functions.

We have

$$\frac{d\mu^2}{dt} = F' \frac{d\theta}{dt} = cu^2 F' \qquad (3.18)$$

If terms of u and its derivatives, this is equivalent to

$$u^2 (u''' + u') = \frac{1}{c^3} \frac{d\mu^2}{dt} \qquad (3.19)$$

In view of the boundedness of u' (β) and U (β) equation (3.19) shows that $\frac{d\mu^2}{dt}$ tends to zero as u tends to zero.

The result obtained by G.N. Dubshin states that the necessary condition for the radius vector to become infinite is that the limit of $\frac{d\mu^2}{dt}$ as t tends to infinity exists and is equal to zero , i.e.,

$$\lim_{t \to \infty} \frac{d\mu^2(t)}{dt} = 0 \qquad (3.20)$$

If $\mu(t)$ satisfies the condition (3.20), then under certain initial conditions r tends to infinity as t tends to infinity. And if $\mu(t)$ does not satisfy this condition, then the radius vector will be restricted for any finite initial values and for any t.

Let us examine an interesting case studies by Stremgren[114] when u^2 is a linear function of time. The osculating Kepler elements

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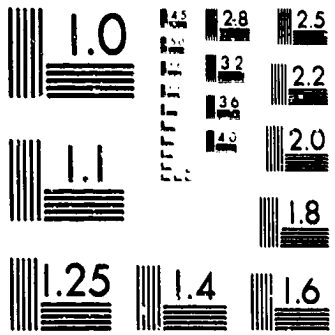

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of orbital motion of a point, divided by μ^2 are examined and disturbances are determined, i.e., the difference between the osculating elements and the constant undisturbed elements, determined at the initial instant. It has been proved, that with such determination of osculating elements, neither the eccentricity nor the longitude of the peri-center contain secular members. Explicit selection of components, quadratic and linear in relation to time are given in formulae for perturbation of the average anomaly.

In the work of Samoilova - Yakhontova[87] Stremgren's method is given for cases when μ^2 is a continuous differential function of time.

The method of analysis by Magnardaze[49 - 51] where recurring equations for calculating the general term of representation in the form of exponential series of time for Cartesian coordinates and its derivatives in Complanar problem of three bodies, one of which has a variable mass and the other two have large masses and move in a pre-set system, are given.

3. Motion under Absolutely Zero Velocities of the Changing Masses.

If the masses of the two heavenly bodies change due to (on account of) cosmic doud found in the statistical equilibrium in respect of certain system of coordinates, it is treated as similar to the problem of the two points whose motion is described by the differential equations

$$m_1 \frac{d^2\bar{r}_1}{dt^2} = - \frac{k^2 m_1 m_2 (\bar{r}_1 - \bar{r}_2)}{|\bar{r}_1 - \bar{r}_2|^3} - \frac{dm_1}{dt} \frac{d\bar{r}_1}{dt} \qquad (3.21)$$

$$m_2 \frac{d^2\bar{r}_2}{dt^2} = - \frac{k^2 m_1 m_2 (\bar{r}_2 - \bar{r}_1)}{|\bar{r}_1 - \bar{r}_2|^3} - \frac{dm_2}{dt} \frac{d\bar{r}_2}{dt} \qquad (3.22)$$

Relative redius vector and relative coordinates (3.3) and the above equations yield the equation of relative motion in the form of

$$\frac{d^2\bar{r}}{dt^2} + \frac{k^2(m_1 + m_2)\bar{r}}{r^3} = \frac{1}{m_1} \frac{dm_1}{dt} \frac{d\bar{r}_1}{dt} - \frac{1}{m_2} \frac{dm_2}{dt} \frac{d\bar{r}_2}{dt} \qquad \ldots\ldots (3.23)$$

Equation (3.21) and (3.22) assume vector integral of momentum.

$$m_1 \frac{d\bar{r}_1}{dt} + m_2 \frac{d\bar{r}_2}{dt} = \bar{A} \qquad (3.24)$$

Here it is not difficult to see that the equations are added. Since

$$\frac{dr_2}{dt} - \frac{dr_1}{dt} = \frac{dr}{dt}.$$

and in view of (3.24), the first part of equation (3.23) can be written as

$$\frac{d\bar{r}}{dt}\ \frac{d}{dt} \ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \frac{\bar{A}}{m_1+m_2}\ \frac{d}{dt} \ln \frac{m_1}{m_2} \qquad (3.25)$$

Suppose that the constant vector momentum $\bar{A}$ is equal to zero or the quantity $\frac{m_1}{m_2}$ remains constant, then the equation (3.23) can be written as

$$\frac{d^2\bar{r}}{dt^2} + \frac{k^2(m_1+m_2)\bar{r}}{r^3} = \frac{d\bar{r}}{dt}\ \frac{d}{dt} \ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad (3.26)$$

Projecting equation(3.26) we get

$$\begin{aligned}
\ddot{x} + \frac{k^2(m_1+m_2)x}{r^3} &= x \frac{d}{dt} \ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right), \\
\ddot{y} + \frac{k^2(m_1+m_2)y}{r^3} &= y \frac{d}{dt} \ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right), \\
\ddot{z} + \frac{k^2(m_1+m_2)z}{r^3} &= z \frac{d}{dt} \ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right).
\end{aligned} \qquad (3.27)$$

Equation (3.27) admits the following three area integrals

$$\begin{aligned} yz - zy &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right) c_1, \\ zx - xz &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right) c_2 \\ xy - yx &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right) c_3 \end{aligned} \qquad (3.28)$$

Therefore, the motion occurs in the plane

$$c_1 x + c_2 y + c_2 z = 0. \qquad (3.29)$$

In this plane the double sectional velocity is equal to

$$2\frac{d\sigma}{dt} = r^2\dot{\theta} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) c, \quad c = \sqrt{c_1^2 + c_2^2 + c_3^2} \qquad (3.30)$$

Projecting the equation (3.27) in the radius vector we get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{k^2(m_1 + m_2)}{r^3} + \dot{r}\frac{d}{dt}\ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad (3.31)$$

Considering the integral, we find

$$r = \frac{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^2 c^2}{r^3} - \frac{k^2(m_1 + m_2)}{r^2} + \dot{r}\frac{d}{dt}\ln\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \qquad \ldots\ldots(3.32)$$

Set

$$u = \frac{1}{r}$$

Then (3.30) gives

$$dt = \frac{d\theta}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right) cu^2} \tag{3.33}$$

Multiplying the equation (3.32) by r^2 yields Benz formula in the form

$$u'' + u = \frac{1}{c^2} f(t) \tag{3.34}$$

where

$$f(t) = \frac{k^2 m_1^2 m_2^2}{m_1 + m_2} \tag{3.35}$$

and u' denotes derivative for θ.

Further suppose that $F(\theta) = f\left[t(\theta)\right]$, then equation (3.34) becomes

$$u'' + u = \frac{1}{c^2} F(\theta) \tag{3.36}$$

Let us suppose that masses m_1 and m_2 are finite (restricted) and not becoming zero in any finite interval $\left[t_o, T\right]$, then from equation (3.32) it follows that $r(t)$ would be finite quantity in any finite interval. If m_1 and m_2 represent single valued and differentiable functions of time then we will see that $f(t)$ is also single valued and differentiable. In view of the relation (3.33)

$F(\theta)$ also turns out to be a single valued function of y the polar angle θ .

Differentiating (3.36) wrt θ_1 and multiplying by u_1^2 we have

$$u^2 (u'' + 2') = \frac{k^2}{c^2} \frac{m_1 m_2}{m_1+m_2} \frac{d}{dt} \frac{m_1^2 m_2^2}{m_1 + m_2} \qquad (3.37)$$

Let θ be a value of θ such that $u(\theta) = 0$. Then the single valued character of u requires the quantities $u'(\theta) + u(\theta)$ to be finite from here, we see that as $u \to 0$.

$$\varphi(t) = \frac{m_1 m_2}{m_1+m_2} \frac{d}{dt} \frac{m_1^2 m_2^2}{m_1 + m_2} \qquad (3.38)$$

also tends to zero.

Thus the existence and equality of the limit of the function $\varphi(t)$, as t tends to infinity, is the necessary condition for the relative radius to acquire infinitely large values.

4. EQUILIBRIUM OF THE BODY HAVING A VARIABLE MASS:

Let us obtain equations of the equilibrium of body having a variable mass and consider some illustrating examples. In particular, we will derive the formulas for the principal vector and the principal torque (moment of force) of the force of free molecular current on the body.

1. Equations of Equilibrium of the Body:

By a body of a variable mass is meant a closed-packed system of the material points of a variable mass, the distance between which does not change. Thus, while examining a body of a variable mass we consider the points with corresponding masses to be represented by small areas of the coordinate system finally connected with the frame of the body. This view preserves the Kinematics of the body having constant mass in the mechanics of a variable mass. Since the fundamental equation 1.2 in chapter I was developed on the strength of he relative motion of the particles and in the sense of a solid body there may be movement of the particles, therefore, we will study their impact on the motion of the body.

Let the body having variable mass be fixed. Then

$$\frac{d^*}{dt}\bar{Q} = 0, \quad \frac{d^*}{dt}\bar{l} = \underset{0}{0}$$

This yields the kinetic moment l of the body in respect of fixed point. In view of the equations (1.8) and (2.4) for the change of the principal vector of the moment and the Kinetic moment of the system of the points having variable mass we will obtain the following vector equations of the equilibrium of the body having a variable mass:

$$\bar{F} + \bar{R} = 0, \quad \bar{L} + \bar{M} = 0. \qquad (4.1)$$

$\bar{F}$ and $\bar{L}$ are principal vector and the principal moment of the external forces and $\bar{R}$ and $\bar{M}$ are the principal vector and the principal moment of the reaction forces.

The auxiliary systems related to the body serve to determine the reaction forces of the points of the body. Therefore, the angular velocity of the rotation of the auxiliary systems is equal to zero and hence the principal vector and the principal moment of Cariliso forces, vanish. The relative accelerations of the particles will be equal to their absolute accelerations.

Equation (4.1) may be rewritten in the form

$$\bar{F} + \bar{\Phi} - \sum m\bar{w} = 0 \qquad (4.2)$$

$$\bar{L} + \bar{K} - \sum \bar{r} \times m\bar{w} = 0. \qquad (4.3)$$

where $\bar{\Phi}$ and $\bar{K}$ denote the principal vector and the principal moment of the impulsive forces. $\bar{W}$ - acceleration of the particles

$$X + \Phi_x - \sum mw_x = 0 \qquad (4.4)$$

$$Y + \Phi_y - \sum mw_y = 0$$

$$Z + \Phi_z - \sum mw_z = 0;$$

$$L_x + K_x - \sum m(yw_z - zw_y) = 0$$

$$L_y + K_y - \sum m(zm_x - xw_z) = 0 \qquad (4.5)$$

$$L_z + K_z - \sum m(zw_y - yw_x) = 0$$

Equation (4.2) and (4.3) as well as t projections (4.4) and (4.5) can be used for determination of reaction during the processes, connected with the movement of particles. Here below the corresponding examples are illustrated:

Example-1:

Take a formula for the draught of air screw as shown in Fig.6. Let v the velocity of the incident stream and u_2 be the velocity with which the particles of air push out from the screw. Denote by x the ux is of the rotation of the screw. We will consider the occurrence of the instantaneous change $u_r - v$ in the velocity of air particles. For this, we take only impulsive force different from zero and the projection of which on the X-axis gives

$$\Phi_x = G_B g^{-1} (u_r - v) \qquad (4.6)$$

where G_B - w ight of air pushing not from the screw per second; g- gravity acceleration.

Put $uv_r^{-1}u$, then in accordance with the first equation of the system (4.4) and equation (4.6) we obtain

$$X = - G_B g^{-1} u_r (1 - v)$$

In this the quantity $G_B g^{-1} u_r$ is called velocity bend or kinetic bend.

Since by third law of Newton, draught was the value X with negative sign, we finally get

$$F = G_B g^{-1} u_r (1 - \nu).$$

Example-2

To determine the horizontal component of the pressure of water set on the fixed paddle of the turbine wheel if the quantitative (volumetric) expenditure of water Q, and specific gravity γ,

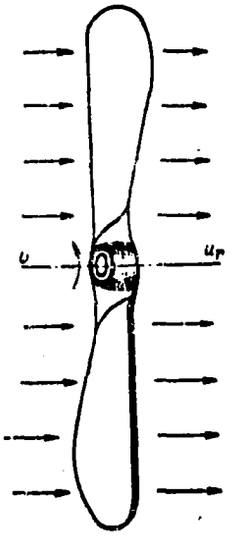

Рис. 6

Fig.6

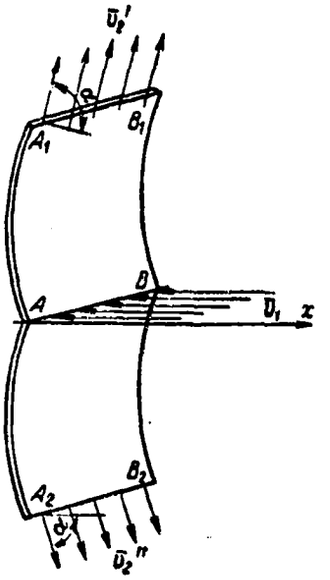

Рис. 7

Fig.7

From Fig.7 it is evident that

$$\left|\bar{v}_2\right| = \left|\bar{v}_2''\right| = v_2$$

In the present case $\Phi = 0$, we have

$$\sum m\bar{w} = \sum m \frac{d\bar{v}}{dt}.$$

Since in this formula, $m = \frac{1}{2}\gamma g^{-1}Qdt$, therefore

$$\sum mw = \frac{1}{2}\frac{\gamma}{g}Q\int_A^{A_1} dv + \frac{1}{2}\frac{\gamma}{g}Q\int_A^{A_1} d\bar{v}$$

from here we obtain

$$\sum m\bar{w} = \frac{1}{2}\frac{\gamma}{g}Q(\bar{v}_2 - \bar{v}_1) + \frac{1}{2}\frac{\gamma}{g}Q(\bar{v}_2 - \bar{v}_1).$$

since when = 0 we have

$$X - \sum mw_x = 0,$$

therefore, we find

$$F_x = -X = -\frac{\gamma}{g}Q(v_1 + v_2\cos\alpha).$$

2. Principal Vector and Principal Moment of the Force of Action resulting from Free-Molecular Flux (Flow) on the Body:

Let a body with convex surface rest on the incident stream of electrically material gas which may flow in a free-molecular state. The problem of flowing free-molecular stream round the body leads

to the case of Meshersky surface interaction with the body of changing particles.

We will represent each point of the surface of the body in three dimensional cartesian system (x_1, x_2, x_3,) such that x_1 is along the external normal and x_2 and x_3 in the tangential plane. We denote by $u_1^{(1)}$, $u_2^{(1)}$ $u_3^{(1)}$ projections in the cartesian system of the velocity of combining the molecules and by $u_1^{(2)}$, $u_2^{(2)}$, $u_3^{(2)}$- the projections of the velocity of seperating the molecules. The free-molecular flux (stream) characterizes a distribution functions f. This function provides [10] the number of molecules d τ found at a given time t in a volume element d τ with coordinates x_1, x_2, x_3 and velocities lying in the interval from $u_1^{(p)}$, $u_2^{(p)}$, $u_3^{(p)}$, to $u_1^{(p)} + du_1^{(p)}$, $u_2^{(p)}$, $u_3^{(p)}$ as follows:

$$dN = \check{n} f\,(x_1, x_2, x_3, u_1^{(p)}, u_2^{(p)}, u_3^{(p)}, t)\, dx_1 dx_2 dx_3 du_1^{(p)} du_2^{(p)} du_3^{(p)}. \quad (4.7)$$

In expression (4.7), n represents the number of particles in a unit volume of gas. (4.7) defines Boltzman equation.

Suppose the gaseous medium moves with a velocity u and is in the thermodynamical equilibrium. When these conditions are present, a solution of the Boltzman equation is identified in Maxwells distribution function

$$f = A_1 e^{-\check{m} h_1 \left[(u_1^{(1)})^2 + (u_2^{(1)} + u_2)^2 + (u_3^{(1)} + u_3)^2\right]}, \quad (4.8)$$

where

$$A_1 = \left(\frac{\check{m} h_1}{\pi}\right)^{\frac{3}{2}}. \quad (4.9)$$

In expressions (4.8) and (4.9)

$\breve{m}$ - mass of one molecule

h_1 - positive constant which is inversely proportional to the absolute temperature T_1 of the gas

u_1, u_2 and u_3 projections of the velocity u of the flux

$$\bar{R} = - \iint_{(S)} \bar{R}^{(S)} \, dS, \qquad (4.10)$$

$$\bar{M} = - \iint_{(S)} \bar{r} \times \bar{R}^{(S)} \, ds \qquad (4.11)$$

where integration is performed over the surface of the body.

The reaction forces causes collision of the molecules and their repulsion. Therefore, we have

$$\bar{R}^{(S)} = \bar{R}_1^{(S)} + \bar{R}_2^{(S)}, \qquad (4.12)$$

in which the components $\bar{R}_1^{(s)}$ and $\bar{R}_2^{(s)}$ are contributed by collision of particles and their repulsion.

Since the axis of x_1 is along the exterior normal, therefore, in view of (4.7) and (4.8) we get

$$\bar{R}_1^{(s)} = \rho A_1 \int_{-\infty}^{0} du_1^{(1)} \int_{-\infty}^{+\infty} du_2^{(1)} \int_{-\infty}^{+\infty} u_1^{(1)} u^{(1)} \cdot e^{-\check{m}h_1 \sum_{i=1} (u_i^{(1)} - u_i)^2} du_3^{(1)}$$

.....(4.13)

The quantity

$$\rho = \check{m}\check{n} \qquad (4.14)$$

is density of the gas flow. The limits of the variable $u_1^{(1)}$ are taken from $-\infty$ to zero because the molecules having positive velocities $u_1^{(1)}$ in the direction of x_1 will not experience collision with the surface of the boey.

Projecting the expression (4.13) on the i th axis ($i = 1,2,3$).

$$R_{1i}^{(s)} = \rho A_1 \int_{-\infty}^{0} du_1^{(1)} \int_{-\infty}^{+\infty} du_2^{(1)} \int_{-\infty}^{+\infty} u_1^{(1)} u_i^{(1)} \cdot e^{-\check{m}h_1 \sum_{i=1}^{3} (u_i^{(1)} u_i)^2} du_3^{(1)}$$

.....(4.15)

Put $u_i^{(1)} = u_1^{(1)}$ integrate

$$R_{11}^{(s)} = \rho A_1 \frac{\pi}{\check{m}h_1} \left[- \frac{u_1}{2\check{m}h_1} \cdot e^{- \check{m}h_1 u_1^2} + \right.$$

$$\left. + \left(u_1^2 + \frac{1}{2\check{m}h_1} \right) \left(\frac{\sqrt{\pi}}{2\sqrt{\check{m}h_1}} + \int_0^{-u_1} e^{-\check{m}h_1 \xi^2} d\xi \right) \right] \qquad (4.16)$$

Taking into consideration the angle ϑ between the x_1 and the opposite direction of the velocity u_1 we have

$$u_1 = u \cos\vartheta \qquad (4.17)$$

The most probable velocity of the thermal motion of the molecule is denoted by C_1 and is determined from the equation

$$c_1 = \frac{1}{\sqrt{\check{m}h_1}} \qquad (4.18)$$

Transforming the equation (4.16) with the help of (4.9), (4.17) and (4.18), we get

$$R_{11}^{(s)} = \frac{1}{2\sqrt{\pi}} \rho c_1 u \cos\theta \cdot e^{- \left(\frac{u \cos\theta}{c_1} \right)^2} +$$

$$+ \rho \left(\frac{c_1^2}{4} + \frac{u^2 \cos^2\theta}{2} \right) \left[1 + \operatorname{erf} \left(\frac{u \cos\theta}{c_1} \right) \right], \qquad (4.19)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta. \qquad (4.20)$$

Directing the axis x_2 along the negative projection of the velocity u of incident stream in the tangent plane, we will have

$$u_2 = - u \sin\theta, \; u_3 = 0 \qquad (4.21)$$

Put $u_i^{(1)} = u_2^{(1)}$ in (4.15) and integrate

$$R_{12}^{(s)} = \rho A_1 \frac{\pi}{\check{m}h_1} u_2^{(1)} \left(- \frac{1}{2\check{m}h_1} e^{- \check{m}h_1 u_1^2} + u_1 \int_0^{-u_1} e^{-\check{m}h_1 \xi^2} d\xi \right) \qquad (4.22)$$

Now in view of (4.9), (4.17), (4.18) and (4.21), the expression (4.22) is transformed into

$$R_{12}^{(s)} = \frac{1}{2\sqrt{\pi}} \rho u c_1 \sin\theta \left[e^{- \left(\frac{u \cos\theta}{c_1} \right)^2} + \frac{\sqrt{\pi}\, u \cos\theta}{c_1} \operatorname{erf} \left(\frac{u \cos\theta}{c_1} \right) \right] \qquad (4.23)$$

In formula (4.15), in place of $u_i^{(1)}$ setting value $u_3^{(1)}$ which vanishes in the chosen coordinate system then we get

$$R_{13}^{(s)} = 0 \qquad (4.24)$$

In case of specular reflection of the molecules

$$R_{21}^{(s)} = R_{11}^{(s)}, \; R_{22}^{(s)} = \overset{R_{12}^{(s)}}{-} R_{23}^{(s)} = - R_{13}^{(s)} \qquad (4.25)$$

The factor $R_{21}^{(s)}$ introduced by the diffuse reflection may be obtained from (4.16) by putting $u_1 = 0$ and replacing h_1 by h_2. The constant h_2 is inversely proportional to the effective temperature

T_2 and is determined by adjustment equation for the wall of the body. From here, we write

$$R_{21}^{(s)} = \frac{1}{4} \rho c_2^2 \tag{4.26}$$

Here

$$c_2 = \frac{1}{\sqrt{\check{m}h_2}} \tag{4.27}$$

characterises the most probable velocity of the diffuse reflection of the molecules.

The number of particles received or repulsed per second is given by

$$\frac{dm_1}{dt} = \check{m}n_1, \quad \frac{dm_i}{dt} = \check{m}n_2. \tag{4.28}$$

where w_1 and w_2 are the corresponding number of the molecules, knocked or reflected in the unit time.

The values of w_1 and w_2 are obtained from the relation (4.7) and (4.8) in the form

$$n_1 = \check{n}A_1 \int_{-\infty}^{0} du_2^{(1)} \int_{-\infty}^{+\infty} du_2^{(1)} \int_{0}^{+\infty} \left| u_1^{(1)} \right| \cdot e^{-mh_1 \sum_{i=1}^{3} (u_i^{(1)} - u_i)^2} du_3^{(1)}, \tag{4.29}$$

$$n_2 = \check{n}A_2 \int_{0}^{+\infty} du_1^{(2)} \int_{-\infty}^{+\infty} du_2^{(2)} \int_{-\infty}^{+\infty} u_1^{(2)} \cdot e^{-\check{m}h_2 \sum_{i=1}^{3} (u_1^{(2)})^2} du_3^{(1)} \tag{4.30}$$

In (4.30),

$$A_2 = \left(\frac{\check{m}h_2}{\pi} \right)^{\frac{3}{2}} \tag{4.31}$$

Performing the integration in (4.29) and (4.30) and making adjustments in the light of (4.9), (4.17), (4.18), (4.27) and (4.31) we get

$$n_1 = \overset{\vee}{n}\left\{\frac{c_1}{2\sqrt{\pi}}\, e^{-\frac{(u\cos\theta)^2}{c_1}} + \frac{u\cos\theta}{2}\left[1 + \operatorname{erf}\left(\frac{u\cos\theta}{c_1}\right)\right]\right\} \tag{4.32}$$

$$n_2 = \overset{\vee}{n}\,\frac{c_2}{2\sqrt{\pi}} \tag{4.33}$$

We will examine the diffuse reflection in detail. Since the mass of the body does not change, therefore $\frac{dm_1}{dt} = \frac{dm_2}{dt}$ and thus, by formula (4.28), $n_1 = n_2$. From (4.32) and (4.33) we find

$$c_2 = c_1 e^{-\left(\frac{u\cos\theta}{c_1}\right)^2} + \sqrt{\pi}\, u\cos\theta\left[1 + \operatorname{erf}\left(\frac{u\cos\theta}{c_1}\right)\right] \tag{4.34}$$

Expressions (4.26) and (4.36), yield

$$R_{21}^{(s)} = R_{\alpha}^{(s)} \tag{4.35}$$

where

$$R_{\alpha}^{(s)} = \frac{1}{4}\rho c_2\left\{c_1 e^{-\left(\frac{u\cos\theta}{c_1}\right)^2} + \sqrt{\pi}\, u\cos\theta\left[1 + \operatorname{erf}\left(\frac{u\cos\theta}{c_1}\right)\right]\right\} \tag{4.36}$$

In the event of the diffuse reflection we thus have

$$R_{22}^{(s)} = 0,\quad R_{23}^{(s)} = 0. \tag{4.37}$$

Now suppose that a part of the molecules σ undergoes the diffuse reflection and the remaining part $1-\sigma$ undergoes the specular reflection, then by virtue of (4.25), (4.35) and (4.37), we have

$$R_{21}^{(s)} = (1 - \sigma)\, R_{11}^{(s)} + \sigma\, R^{(s)}, \tag{4.38}$$

$$R_{22}^{(s)} = -(1 - \sigma)\, R_{12}^{(s)}$$

$$R_{23}^{(s)} = 0.$$

For the projections of the vector $R^{(s)}$, in view of (4.12), (4.24) and (4.38), we get

$$R_{x_1}^{(s)} = (2 - \sigma)\, R_{11}^{(s)} + \sigma\, R^{(s)}, \tag{4.39}$$

$$R_{x_2}^{(s)} = \sigma\, R_{12}^{(s)},\quad R_{x_3}^{(s)} = 0$$

The projections of the vector $R^{(s)}$, using (4.19), (4.23), (4.36) and (4.39), finally takes the form

$$R_{x_1}^{(s)} = \frac{\rho u^2}{2}\left\{(2-\sigma)\,\frac{\cos\theta}{s\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2z}(1 + \operatorname{erf} z) + X(z)\right] + \frac{\sigma}{2s^2}\cdot\frac{c}{c_1}\,X(z)\right\}, \tag{4.40}$$

$$R_{x_2}^{(s)} = \frac{\rho u^2}{2}\,\frac{\sigma \sin}{s\sqrt{\pi}}\,X(z),\quad R_{x_3}^{(s)} = 0. \tag{4.41}$$

Here $X(z) = e^{-z^2} + \sqrt{\pi}\, z(1 + \operatorname{erf} z)$, $z = s\cos\theta$, $s = \frac{u}{c_1}$ (4.42)

In developing the expressions (4.40) and (4.41) the teansformations from the works of R.G. Barantseva[4] were used. In

his works, the formulas of the form (4.40) and (4.41) are employed for the calculation of aerodynamical coefficients of the translational motion for rough bodies of simplest form in a free -molecular stress (flux).

Let us consider the application of (4.40) and (4.41) in finding the projections of the moment of reaction forces about the axis (x,y,z) finally connected with the body. Let $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$ be the unit vectors of the axes (x_1, x_2, x_3). Since $R^{(s)}_{x3} = 0$, then from (4.11) we have

$$M_x = -\iint_{(s)} \left[y(e_1 R^{(s)}_{x_1} + e_{2z} R^{(s)}_{x_2}) - z(e_{1y}R^{(s)}_{x_1} + e_{2y}R^{(s)}_{x_2}) \right] ds,$$

$$M_y = -\iint_{(s)} \left[z(e_{1x}R^{(s)}_{x_1} + e_{2x}R^{(s)}_{x_2}) - x(e_{1z}R^{(s)}_{x_1} + e_{2z}R^{(s)}_{x_2}) \right] ds, \quad (4.43)$$

$$M_z = -\iint_{(s)} \left[x(e_{1y}R^{(s)}_{x_1} + e_{2y}(R^{(s)}_{x_2}) - y(e_{1x}R^{(s)}_{x_1} + e_{2x}R^{(s)}_{x_2}) \right] ds.$$

Here e_{1x}, e_{1y} and e_{iz} are directions cosines of the axis x_1; e_{2x}, e_{2y} and e_{2z} are directions cosins of the axis x_2 in the system (x,y,z) are variable coordinates of a point on the surface of the body.

Let us examine a truncated circular core with a semi-solid angle β and having Z-axis along the symmetric axis of the core. Let a be the radius of the cross section passing through the center of the mass. Place the origin of the system (x,y,z) at the center of the mass. Let z_1 and z_2 be coordinates for points located in the upper and lower sections of the base, respectively; $z_1 < 0$ and $z_2 > 0$. Let us presume that in this case, the diameter of the lower section of the

base is less than the diameter of the upper section. Now let us examine the distance h, that equals $h = a\,\mathrm{ctg}\,\beta$.

The angle between the axis x and projection x_1 for the surface x,y, we now mark γ . We have obtained for the lateral surface, $e^{б}_{1x} = \cos\beta\cos\gamma$, $e^{б}_{1y} = \mathrm{cost}\beta\sin\gamma$, $e^{б}_{1z} = \sin\beta$; for the upper base: $e^{в}_{1x} = 0$, $e^{в}_{1y} = 0$, $e^{в}_{1z} = -1$ and for the lower base: $e^{н}_{1x} = 0$, $e^{н}_{1y} = 0$, $e^{н}_{1z} = 1$.

The directing cosines of axis x_2 are determined with the help of the following equation system:

$$\bar{e}_2(\bar{u}\times\bar{e}_1) = 0,\ \bar{e}_1\bar{e}_2 = 0.\quad e^2_{2x} + e^2_{2y} + e^2_{2z} = 1;\ \bar{u}\bar{e}_2 < 0 \qquad (4.44)$$

Let us fine the solution to equation (4.44), that corresponds to the lateral surface, upper and lower bases. For the lateral surface we have:

$$e_{2x} = \varepsilon_б A_б C^{-1},\ e_{2y} = \varepsilon_б B_б C^{-1}_б,$$

$$e_{2z} = \varepsilon_б C^{-1}_б,\ \varepsilon_б = \pm 1,\ C_б = \sqrt{1 + A^2_б + B^2_б} \qquad (4.45)$$

$$\varepsilon_б(a_{11}A_б + a_{12}B_б + a_{13}) > 0$$

Here we have taken symbols:

$$A_б = \frac{b_б f_б - c_б e_б}{a_б e_б - b_б d_б},\quad B_б = \frac{d_б c_б - a_б f_б}{d_б e_б - b_б d_б}$$

$$a_б = a_{12}\sin\beta + a_{13}\cos\beta\sin\gamma,$$

$$b_б = a_{11}\sin\beta + a_{13}\cos\beta\cos\gamma,$$

$$c_б = a_{11}\cos\beta\sin б - a_{12}\cos\beta\cos\gamma,$$

$$d_б = \cos\beta\cos\gamma,\ e_б = \cos\beta\sin\gamma,\ f_б = \sin\gamma.$$

where a_{11}, a_{12}, a_{13}, - directing cosines for the system (x,y,z) for reverse direction of velocity u of inflowing gas.

For the upper and lower base we shall record:

$$e^{ОСН}_{2x} = \varepsilon_{ОСН}A_{ОСН}C^{-1}_{ОСН},\ e^{ОСН}_{2y} = \varepsilon_{ОСН}C^{-1}_{ОСН},\ e^{ОСН}_{2z} = 0$$

Changing in formulae (4.43) from the surface to multiple integrals, we get:

$$M_x = -\operatorname{tg}\beta \int_0^{2\pi} R^{(s)}_{x_1 6}\,(A\operatorname{tg}^2\beta - B)\sin\gamma + R^{(s)}_{2z_6}\operatorname{tg}\ \sin\ - Be^{6}_{2y})\ d\ +$$

$$+ \frac{\pi}{2}\, e^{B}_{2y}\left[R^{(s)}_{x_2 B} z_1\,(a + z_1\operatorname{tg}\beta) + R^{(s)}_{x_2 H}\, z_2\,(a + z_2\operatorname{tg}\beta)\right] \qquad (4.47)$$

$$M_y = -\operatorname{tg}\beta \int_0^{2\pi}\left[R^{(s)}_{x_1 6}\,(A\operatorname{tg}^2\beta - B)\cos\gamma + R^{(s)}_{x_2 6}\,(Ae^{6}_{2y}\operatorname{tg}\beta\sin\gamma - Be^{6}_{2x})\right]d\gamma$$

$$+ \frac{\pi}{2}\, e^{B}_{2x}\left[R^{(s)}_{x_2 B} z_1\,(a + z_1\operatorname{tg}\beta) + R^{(s)}_{x_2 H} z_2\operatorname{tg}\beta)\right], \qquad (4.48)$$

$$M_z = -A\,\frac{\operatorname{tg}^2\beta}{\cos\beta}\int_0^{2\pi} R^{(s)}_{x_2 6}\,(e^{B}_{x_2}\cos\gamma - e^{H}_{x_1}\sin\gamma)\,d\gamma. \qquad (4.49)$$

Here

$$A = h^2(z_1 + z_2) + h(z_1^2 - z_2^2) + \frac{1}{3}(z_1^3 + z_2^3),$$

$$B = \frac{1}{2}h(z_1^2 - z_2^2) + \frac{1}{3}(z_1^3 + z_2^3) \qquad (4.50$$

Indexes ,B,H indicate that the corresponding values are determined for the lateral, upper and lower surfaces.

Formulae (4.47) - (4.49) are useful for finding the moments of interactions of a free-molecular stress on the body having the shape of truncated core. It may be remarked that it is not difficult to develop similar formulae for the other convex solids of revolution.

5. TRANSLATIONAL MOTION OF THE BODY HAVING A VARIABLE MASS:

1. Equation of the Translational Motion of the Body Having a Variable Mass:

In the last paragraph it was remarked that the motion of a body having variable mass may be viewed as motion of a fixed system, finally connected with the solid body referred to above.

In the event of translational motion all points of the body have the same accelerations $\bar{w}$. Therefore, we get

$$\frac{d^2\bar{r}_C}{dt^2} = w.$$

In case of an auxiliary system for all points of the body we consider a fixed coordinate system which us unalterably associated with the body. Therefore,

$$\bar{J} = 0$$

Equation (1.16) becomes

$$m\bar{w} = \bar{F} + \bar{R} \tag{5.1}$$

Where

$$\bar{R} = -\sum \bar{m}\bar{w}_r + \Phi \tag{5.2}$$

In this connection we will discuss the example of the reactive boat. I.V. Meshersky explains (examins) the translational motion of the boat. (Fig.8)

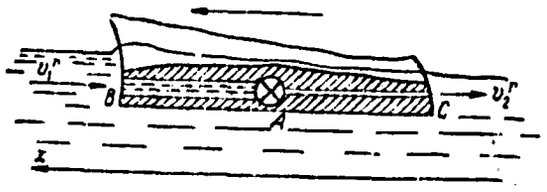

Рис. 8

Fig.8

The boat is operated by a pump A which lifts water from the entrance channel BA to the exit channel AC. Let v_1^r and v_2^r be the relative velocities of the incoming and outgoing water. Meshersky thinks that these velocities are independent of the velocity of the boat.

Denote by p the mass of the water entering the boat through the channel in the units time and by v the ratio of the area of the entrance aperture and the area of the exit aperture, then we get

$$\frac{dm_1}{dt} = pv_1^r, \quad \frac{dm_2}{dt} = \frac{1}{n} pv_2^r, \quad v_2^r = nv_1^r.$$

Assuming the resistance to be propertional to the second degree of the velocity - k^2x^2,. Meshersky obtains

$$mx = - k^2x^2 + p (v_1^r)^2 (n - 1) \qquad (5.3)$$

Here m represents the weight of the boat together with the water in it. It will be a constant quantity. We can, of course, reject the pressure

of the water flowing behind the boat.

Meshersky equation (5.3) explains the motion both for constant and variable velocity v_1^r as well as for the accelerated function of the pump.

We obtain the equation of motion of the boat from the equation (5.1) in the form

$$mx = k^2x^2 + p\,(v_1^r)^2\,(n-1) + w_1^r\,(m_{BX} + nm_{BMX}) \quad (5.4)$$

Where m_{BX} is mass of the water in entrance channel BA, w_1^r is its relative acceleration; m_{BUX} is mass of the water in the exit channel AC.

The equation (5.4) reduces to equation (5.3) given by Meshersky when $w_1^r = 0$ i.e., when the pump operates throughout with uniform efficiency (with the same speed). From equation (5.3) it is concluded that if $n = 1$, the boat will not then move, whereas equation (5.4) shows that the motion is possible under some conditions.

2. Examples of Translational motion of the Body having a Variable Mass and without Internal Movement of Particles:

When there is no relative motion of particles (or when the particles move with uniform relative velocities) we get

$$\bar{w}_r = 0, \quad \bar{R} = \Phi$$

Suppose the particles do not move with uniform relative velocities,

then the impulsive forces are different from zero (do not vanish) only in respect of points on the boundary of the solid body. Hence, in the case of formulae (1.11) and (1.12), used for the calculation of the principal vector of impulsive forces $\bar{\Phi}$, integration and seen motion may be carried over only on the points lying on the surface of the body having a variable mass.

In the present case, the impulsive force j-th surface point, taking into consideration section of chapter 1, may be written in the form

$$\bar{\Phi}_j = \frac{dm_{1j}}{dt}(\bar{u}_{1j} - \bar{v}_j) - \frac{dm_{2j}}{dt}(\bar{u}_{2j} - \bar{v}_j), \qquad (5.5).....$$

where, $\frac{dm_{1j}}{dt}$ represents the rate of gaining mass by the j-th point; u_{1j} represents the velocity of the center of inertia of these masses; $\frac{dm_{2j}}{dt}$ represents the rate of reduction of the mass of j-th point, u_{2j} represents, the velocity of the inertial center of the reduced mass, v_j is the velocity of j-th point of the solid body. Mass of j-th point is equal to

$$m_j = m_{oj} + m_{1j} - m_{2j},$$

where m_{oj} is a constant quantity and m_{1j} and m_{2j} represent the non-negative functions characterizing the addition and reduction of the mass.

When exclusively one of the processes of addition or ejection of particles takes place in j-th point, formula (5.5) takes the form

$$\bar{\Phi}_j = \frac{dm_j}{dt}(\bar{u}_j - \bar{v}_j). \qquad (5.6).....$$

In (5.6), u_j character the velocity of the inertial center of the additions or substraction of the mass.

In such a case, the principal vector of the impulsive forces is equal to

$$\bar{\Phi} = \sum_{j} \frac{dm_j}{dt} (\bar{u}_j - \bar{v}_j), \qquad (5.7)$$

Where the summation is carried over the surface points of the body. Or

$$\Phi = \int\int_{(s)} (\bar{u} - \bar{v}) \frac{\partial \nu}{\partial t} ds \qquad (5.8)$$

Where (s) - surface enclosing the body, ds - element of the surface referred to, v (x,y,z,t) represents the density of the surface layer mass of which changes.

We will study two examples discussed by I.V. Meshersky[53].

Problem No.1

To determine the vertical use of a fastened balloon assuming that the resistance of air is proportional to the square of velocity ($- k^2x^2$). Let the initial velocity of the balloon on the earth be zero and the lifting force (P) the spindle on which the thread is rolled, is revolved with an angular velocity so that the unrolled (part of the) thread at any instant has the velocity equal to the velocity of the balloon.

The system comprising the balloon and the thread which moves vertically upward with the balloon has mass

$$m = m_o (1 + \alpha x),$$

Where m_o and αx are positive constants and x represents the vertical coordinate of the balloon. The system referred to above

with the nature of the motion is similar to the solid body having a variable mass.

The reaction force arising from the absence of the relative motion of the particles is the same impulsive force which, in the given conditions of the problem, is equal to zero, because the particles of the thread get attached to the mechanical system with a zero relative velocity. Therefore, $\bar{R} = 0$ and the equation (5.1) on projection on the verticles axis Ox becomes:

$$m\ddot{x} = = - mg - k^2\dot{x}^2$$

Or

$$\frac{dx^2}{dx} + \frac{2k^2}{m_o (1 + \alpha x)} x^2 = \frac{2P}{m_o (1 + \alpha x)} - 2g$$

$$\dot{x}^2 = \frac{P}{k^2} - \frac{2g}{\alpha(\varepsilon + 1)} (1 + \alpha x) + c (1 + \alpha x)^{-\varepsilon}, \quad \varepsilon = \frac{2k^2}{\alpha m_o}$$

The initial conditions being $t = 0$, $x_o = 0$, $\dot{x}_o = 0$ constant C is found to be

$$C = \frac{2g}{\alpha(\varepsilon + 1)} - \frac{P}{k^2}$$

PROBLEM No.2 (65)

Suppose a heavy and absolutely flexible chain rests on a table and only one of its links hangs down, as shown in Fig.9. The problem is to find the time taken by the chain in falling down from

the table to the floor. Let the height of the table be b and the length of the chain be a. The process will be accomplished in three stages: first, from the commencement of the motion till the chain touches the floor, called the plane of the floor Q). Secondly from the moment of its contact with the floor till the last link of the chain leaves the table, plane of the table P). And finally, from the moment the last link descends till the entire chain rests on the floor.

In the first stage of the motion, the force of gravity and γx acts on the descending part of the chain, where γ represents weight per unit length of the chain and x is the length of the descending part. [3]

The descending part of the chain acts as a body possessing the translational motion with mass $m = \frac{\gamma}{g} x$, and in which there is no relative motion of the particles. Therefore, the principal vector of the reaction forces is equal o the sum of the impulsive forces which are different from zero at a point where new links join. A link before joining the descending part engaged in the motion rests on the table. When it joins, it joins with jerk and then the link will have the same velocity as that of the solid body under consideration.

Consequently, by (5.6) the principal vector of the impulsive forces on projection on X-ray is that which is directed verticles dowawards is equal to

$$\frac{dm}{dt}(u_x - v_x) = -\frac{\gamma}{g}\dot{x}^2.$$

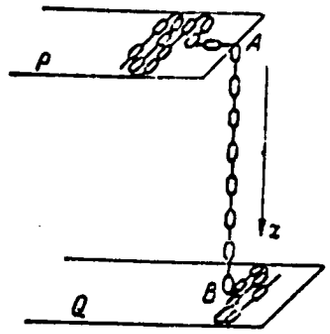

Рис. 9

Fig.9

Equation of the motion (5.1) on the projection on X-ray becomes

$$\frac{\gamma}{g}\,\ddot{x}x = \gamma x - \frac{\gamma}{g}\,\dot{x}^2 ,$$

so, we have

$$\ddot{x}x = gx - \dot{x}^2$$

Since the equality

$$\ddot{x} = \frac{1}{2}\,\frac{d\dot{x}^2}{dx} ,$$

holds therefore,

$$x\,\frac{d\dot{x}^2}{dx} + 2\dot{x}^2 = 2gx.$$

Multiplying by x_1 we get

$$x^2\,\frac{d\dot{x}^2}{dx} + 2x\dot{x}^2 = 2gx^2,$$

$$d\,(x^2\dot{x}^2) = 2gx^2dx,$$

$$x^2\dot{x}^2 = \frac{2}{3}\,gx^3 + C_1$$

Since initially the chain was at rest, therefore for $t_o = 0$, $x_o = 0$. Hence $C_1 = \frac{2}{3} g x_o^2$. The quantity x_o is very small because initially only link of the chain was hanging from the table. Therefore, we take $x_o = 0$ and thus, $C_1 = 0$. Divide both sides by x^2 and take square root, we have

$$\dot{x} = \sqrt{\frac{2}{3} g x}.$$

Integrating once more and taking into consideration the initial conditions, we get

$$x = \frac{1}{6} g t^2$$

Time taken in the first stage motion is

$$t_1 = \sqrt{\frac{6h}{g}} \qquad (5.9)$$

In the second stage of motion, the mass of the moving part of the chain remains constant at $m = \frac{\gamma}{g} h$. The reaction force is the same as in the first stage, because the chain is absolutely flexible and the links of the chain after being deposited on the floor do not produce reaction force. Other facts of the situation have already been discussed in the example given in section 2 of the present chapter.

Therefore, the equation of the motion described in the second stage can be expressed as

$$\frac{\gamma}{g} h\ddot{x} = \gamma h - \frac{\gamma}{g} \dot{x}^2$$

simplifying

$$h\ddot{x} = gh - x^2$$

From here, it follows that

$$\frac{h\dot{dx}}{gh - x^2} = dt.$$

Integrating

$$\frac{1}{2}\sqrt{\frac{h}{g}}\ln\frac{\sqrt{gh} + \dot{x}}{\sqrt{gh} - \dot{x}} = t + C_2. \quad C_2 = \text{const.}$$

For the second stage motion the initial conditions are: when $t_o = 0$:

$$x_o = \sqrt{\frac{2}{3}gh}, \; \dot{x}_o = 0$$

therefore,

$$C_2 = \sqrt{\frac{h}{g}}\ln(\sqrt{3} + \sqrt{2}),$$

$$\dot{x} = \sqrt{gh}\,\frac{(\sqrt{3}+\sqrt{2})e^{\sqrt{\frac{g}{h}}t} - (\sqrt{3}-\sqrt{2})e^{-\sqrt{\frac{g}{h}}t}}{(\sqrt{3}+\sqrt{2})\,e^{\sqrt{\frac{g}{h}}t} + (\sqrt{3}+\sqrt{2})\,e^{-\sqrt{\frac{g}{h}}t}} \qquad (5.10)$$

Integrating and using the initial conditions, we find

$$x = h\ln\frac{1}{2\sqrt{3}}\left[(\sqrt{3}+\sqrt{2})e^{\sqrt{\frac{g}{h}}t} + (\sqrt{3}-\sqrt{2})\,e^{-\sqrt{\frac{g}{h}}t}\right]$$

Suppose $x = a = h$, time taken to complete the second stage motion is calculated to be

$$t_2 \simeq \sqrt{\frac{h}{g}}\left[\frac{a-h}{h} + \ln \frac{\sqrt{3}+\sqrt{3-e^{-\frac{2}{h}(a-h)}}}{\sqrt{3}+\sqrt{2}}\right] \qquad (5.11)$$

In the third stage, mass of the moving chain is $m = \frac{\gamma}{g}(h-x)$ reaction forces is equal to zero. Equation of the motion is obtained in the form

$$\frac{\gamma}{g}(h-x)\ddot{x} = \gamma(h-h),$$

$$\ddot{x} = g.$$

Therefore,

$$x = \dot{x}_o t + \frac{gt^2}{2}.$$

Here x_o is given by the equation (5.10) for the time t_2. Obviously

$$\dot{x}_o = \sqrt{gh}\sqrt{1-\frac{1}{3}e^{-\frac{2}{h}(a-h)}}$$

Time consumed in the third stage motion is determined from the quadratic equation

$$h = \dot{x}_o t_3 + \frac{1}{2}gt_3^2$$

$$t_3 = \frac{1}{g}\left(\sqrt{\dot{x}_3^2 + 2gh} - \dot{x}_o\right).$$

$$t_3 = \frac{1}{\sqrt{3}}\sqrt{\frac{h}{g}}\left[\sqrt{9-e^{-\frac{2}{h}(a-h)}} - \sqrt{3-e^{-\frac{2}{h}(a-h)}}\right] \qquad (5.12)$$

Thus the time t taken by the chain in falling from the table to the floor is

$$t = t_1+t_2+t_3$$

6. EQUATION OF THE MOTION OF THE CENTER OF THE MASS AND THE EULERS EQUATION FOR A BODY HAVING A VARIABLE MASS:

1. Equation of the Motion of Center of Mass:

It was observed in 5 that for a solid body having a variable mass the geometrical points of fixed system firmly connected with the frame of the body represent the points of the system. Denote by v_c^{-e} and w_c^{-e} the (enduring) velocity and the acceleration of the center of mass of the body. They indicate the velocity and the acceleration of that point of the fixed system which identify the center of the mass at a given time.

According to Euler's formula of velocity of points of solid body, for j-th points we have

$$\bar{v}_j = \bar{v}_c^{e} + \bar{\omega} \times (\bar{r}_j - \bar{r}_c) \qquad (6.1)$$

Here $\bar{\omega}$ - angular velocity of the rotation of the body $\bar{r}_j$ - radius vector of j th point and r_c - radius vector of the center of inertia

Since by (1.15) we have

$$\frac{d^*}{dt} r_c = \sum_j m_j \bar{v}_j, \quad \frac{d^{2*}}{dt^2} \bar{r}_c = \sum_j m_j \bar{w}_j,$$

therefore, for a body of variable mass we get

$$\frac{d^*}{dt} \bar{r}_c = \bar{v}_c^e, \quad \frac{d^{2*}}{dt^2} \bar{r}_c = \bar{w}_c^e \tag{6.2}$$

Therefore, the equation of the motion of the inertial center (1.16) is expressed as

$$m \bar{w}_c^e = \bar{F} + \bar{R} \tag{6.3}$$

If the position of the inertial center of the body does not change, then $\bar{w}_c^e = \bar{w}_c$, where w_c indicates the acceleration of the inertial center. Equation (6.3) will then be written as

$$m \bar{w}_c = \bar{F} + \bar{R}. \tag{6.4}$$

Thus equation (6.3) and (6.4) describe the motion of the inertial center of the body having a variable mass.

Let a certain geometrical point A of the body be taken as plus. Then according to Euler's formula we write:

$$\bar{v}_c^e = \bar{v}_A + \bar{\omega} \times (\bar{r}_c - \bar{r}_A) \tag{6.5}$$

Where $\bar{v}_A$ - velocity of the point A, $\bar{\omega}$ - angular velocity of the body, $\bar{r}_c$, $\bar{r}_A$, - radius vectors of the inertial center and the point A

report of the origin of the fixed system of coordinates.

Computing the derivative $\frac{d^*}{dt}$ of left and right hand sides of (6.5) we get

$$\bar{w}_C^* = \bar{w}_A + \dot{\bar{\omega}} \times (\bar{r}_C - \bar{r}_A) + \bar{\omega} \times (\bar{v}_C^* - \bar{v}_A)$$

On account of (6.5) we have

$$\bar{w}_C^* = \bar{w}_A + \dot{\bar{\omega}} \times (\bar{r}_C - \bar{r}_A) + \bar{\omega} \times \left[\bar{\omega} \times (\bar{r}_C - \bar{r}_A) \right.$$

and by simplifying we get

$$\bar{w}_C^* = \bar{w}_A + \dot{\bar{\omega}} \times (\bar{r}_C - \bar{r}_A) + \left[\bar{\omega} (\bar{r}_C - \bar{r}_A) \right] \bar{\omega} - \omega^2 (\bar{r}_C - \bar{r}_A) \qquad (6.6)$$

Taking into consideration the equation of the motion of the inertial center (6.3) we find the equation of the motion of the point A of the body having a variable mass in the following form:

$$\bar{w}_A + \dot{\bar{\omega}} \times (\bar{r}_C - \bar{r}_A) = \frac{1}{m} \bar{F} + \frac{1}{m} \bar{R} + \omega^2 (\bar{r}_C - \bar{r}_A) -$$

$$- \left[\bar{\omega} (\bar{r}_C - \bar{r}_A) \right] \bar{\omega} \qquad (6.7)$$

Let the general coordinates of the fixed system be (ξ, η, ζ). Denote the coordinates of the point A for this system by (ξ, η, ζ) and those of w by ξ_A, η_A, ζ_A; Let $e\xi$, $e\eta$ and $e\zeta$ - be the unit vectors of the fixed system.

We write

$$\dot{\bar{\omega}} \times (\bar{r}_C - \bar{r}_A) = \begin{vmatrix} \bar{e}_\xi & \bar{e}_\eta & \bar{e}_\zeta \\ \omega_\xi & \omega_\eta & \omega_\zeta \\ \xi_C - \xi_A & \eta_C - \eta_A & \zeta_C - \zeta_A \end{vmatrix} =$$

$$= \left[(\zeta_C - \zeta_A)\dot{\omega}_\eta - (\eta_C - \eta_A)\dot{\omega}_\zeta \right] \bar{e}_\xi + \left[(\xi_C - \xi_A)\dot{\omega}_\zeta - \right.$$

$$\left. - (\zeta_C - \zeta_A)\dot{\omega}_\xi \right] \bar{e}_\eta + \left[(\eta_C - \eta_A)\dot{\omega}_\xi - (\xi_C - \xi_A)\dot{\omega}_\eta \right] \bar{e}_\zeta$$

The vectorial equation (6.7) on the projection on the axis of the fixed system are, therefore, obtained as

$$\begin{aligned}
&\ddot{\xi}_A + (\zeta_C - \zeta_A)\dot{\omega}_\eta - (\eta_C - \eta_A)\dot{\omega}_\zeta = \frac{1}{m} F_\xi + \frac{1}{m} R_\xi + \\
&+ (\omega_\eta^2 + \omega_\zeta^2)(\xi_C - \xi_A) - \omega_\xi \omega_\eta (\eta_C - \eta_A) - \omega_\xi \omega_\zeta (\zeta_C - \zeta_A), \\
&\ddot{\eta}_A + (\ddot{\xi}_C - \ddot{\xi}_A)\dot{\omega}_\zeta - (\zeta_C - \zeta_A)\dot{\omega}_\xi = \frac{1}{m} F_\eta + \frac{1}{m} R_\eta + \\
&+ (\omega_\xi^2 + \omega_\zeta^2)(\eta_C - \eta_A) - \omega_\eta \omega_\zeta (\zeta_C - \zeta_A) - \omega_\eta \omega_\xi (\xi_C - \xi_A), \\
&\ddot{\zeta}_A + (\eta_C - \eta_A)\dot{\omega}_\xi - (\xi_C - \xi_A)\dot{\omega}_\eta = \frac{1}{m} F_\zeta + \frac{1}{m} R_\zeta + \\
&+ (\omega_\xi^2 + \omega_\eta^2)(\zeta_C - \zeta_A) - \omega_\zeta \omega_\xi (\xi_C - \xi_A) - \omega_\zeta \omega_\eta (\eta_C - \eta_A).
\end{aligned} \tag{6.8}$$

Take the point A as the origin of the system of coordinates, invariably connected with the body having a variable mass. Denote the projections of the velocity of the point A on these axis by w_x, w_y, w_z respectively, and the projections of the angular

velocity w of the body by w_x, w_y, w_z. The coordinates of the inertial center in this system will be written as x_C, y_C, z_C respectively. We thus obtain

$$\bar{w}_A = \frac{d^* \bar{r}_A}{dt} + \bar{\omega} \times v_A, \quad \frac{d\bar{\omega}}{dt} = \frac{d'\bar{\omega}}{dt},$$

where $\frac{d'}{dt}$ is a relative derivative. Equation (6.7) then becomes

$$\frac{d'\bar{v}_A}{dt} + \frac{d'\bar{\omega}}{dt} \times (r_C - r_A) = \frac{1}{m}\bar{F} + \frac{1}{m}\bar{R} - \bar{\omega} \times \bar{v}_A + \omega^2 (\bar{r}_C - \bar{r}_A) - \left[\bar{\omega}(\bar{r}_C - \bar{r}_A)\right]\bar{\omega} \tag{6.9}$$

By projecting on the axes of the system referred to we have

$$\begin{aligned}
&\dot{\alpha}_A + z_C \dot{\omega}_y - y_C \dot{\omega}_z = \frac{1}{m} X + \frac{1}{m} R_x + \dot{\omega}_z \beta_A - \omega_y \gamma_A + (\omega_y^2 + \omega_z^2) x_C - \\
&\qquad - \omega_x \omega_y y_C - \omega_x \omega_z z_C, \\
&\beta_A + x_C \dot{\omega}_z - z_C \dot{\omega}_x = \frac{1}{m} Y + \frac{1}{m} R_y + \dot{\omega}_x \gamma_A - \omega_z \alpha_A + \\
&\qquad + (\omega_x^2 + \omega_z^2) y_C - \omega_y \omega_z z_C - \omega_x \omega_y x_C, \\
&\dot{\gamma}_A + y_C \dot{\omega}_x - x_C \dot{\omega}_y = \frac{1}{m} Z + \frac{1}{m} R_z + \dot{\omega}_y \alpha_A - \omega_x \beta_A + \\
&\qquad + (\omega_x^2 + \omega_y^2) z_C - \omega_x \omega_z x_C - \omega_y \omega_z y_C.
\end{aligned} \tag{6.10}$$

If $\bar{R} = 0$, then the equations (6.7) - (6.10) are identical with the corresponding equations holding for body with a constant mass.

2. Euler's Equation for Rotation of the Body Having a Variable Mass about a Fixed Point.

A law providing a change in the kinetic moment for the system of points having a variable mass has been worked out in the form of

$$\frac{d^*}{dt}\bar{l} = \bar{L} + \bar{M},$$

where the principal moment of reaction forces is computed with the help of

$$\bar{M} = -\sum \bar{r} \times m \bar{w}_r + \bar{H} + \bar{K}.$$

Here H is the principal moment of the Cariliso force and $\bar{K}$ is the principal moment of impulsive forces. In the formulae referred to above the moments for a fixed point are calculated.

Suppose the body having a variable mass moves in such a manner that the position of the geometrical point O does not change during the entire movement. Take this point as the origin of the Descartes system of the coordinates invariably associated with the body. Thus for a body having a variable mass we have

$$\bar{l} = I\bar{\omega} \qquad (6.11)$$

where, $\bar{w}$ is, as before, the angular velocity of the rotation of the body

$$I = \begin{Vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{Vmatrix}$$

is the tensor of the inertia. In contract to the body with constant mass, the axial moments, of the inertia I_x, I_y and I_y and the centrifugal moments of the inertia I_{xy}, I_{xz}, I_{yz} in respect of a body having a variable quantities.

From (6.11) we see that the projections of the kinetic moment $\bar{I}$ on the x,y,z axes are respectively equal to

$$\begin{aligned} l_x &= I_x \omega_x - I_{xy} \omega_y - I_{xz} \omega_z, \\ l_y &= -I_{xy} \omega_x + I_y \omega_y - I_{yz} \omega_z, \\ l_z &= -I_{xz} \omega_x - I_{yz} \omega_y + I_z \omega_z, \end{aligned} \tag{6.12}$$

where, as before, ω_x, ω_y, ω_z are the projections of the angular velocity $\bar{\omega}$ on x,y,z axes.

On the basis of (2.4) we have

$$\frac{d^*}{dt} (I\bar{\omega}) = \bar{L} + \bar{M} \tag{6.13}$$

Since the equations

$$\frac{d^*}{dt} (I\bar{\omega}) = \frac{d^*}{dt} I\bar{\omega}) + \bar{\omega} \times I\bar{\omega}, \quad \frac{d'^*}{dt} (I\bar{\omega}) = I \frac{d'\bar{\omega}}{dt'}$$

holds, therefore, the vector equation (6.13) is expressed as

$$I\frac{d\bar{\omega}}{dt} + \bar{\omega}_x I\bar{\omega} = \bar{L} + \bar{M} \qquad (6.14)$$

Here the notation with a dash sign denotes the corresponding relative derivative in the system (x,y,z).

Projecting the equation (6.14) on the axes, we get Euler's equations for the rotation of the body having a variable mass about a fixed point.

$$I_x\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z + \omega_y l_z - \omega_z l_z = L_x + M_x$$

$$- I_{xy}\dot{\omega}_x + I_y\dot{\omega}_y - I_{yz}\dot{\omega}_z + \omega_z l_x - \omega_x l_z = L_y + M_y,$$

$$- I_{xz}\dot{\omega}_x - I_{yz}\omega_y + I_z\dot{\omega}_z + \omega_x l_y - \omega_y l_x = L_z + M_z.$$

where l_x, l_y and l_z are determined from the relations (6.12). Taking them into consideration, Euler's equation is finally obtained in the form

$$I_x\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z + (I_z - I_y)\omega_y\omega_z - I_{xz}\omega_x\omega_y + I_{xy}\omega_x\omega_z -$$

$$- I_{yz}(\omega_y^2 - \omega_z^2) = L_x + M_x,$$

$$- I_{xy}\dot{\omega}_x + I_y\dot{\omega}_y - I_{yz}\dot{\omega}_z + (I_x - I_z)\omega_x\omega_z - I_{xy}\omega_y\omega_z + I_{yz}\omega_x\omega_y -$$

$$- I_{xz}(\omega_z^2 - \omega_x^2) = L_y + M_y, \qquad \ldots (6.15)$$

$$- I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y - I_{yz}\omega_x\omega_z - I_{xz}\omega_y\omega_z -$$

$$- I_{xy}(\omega_x^2 - \omega_y) = L_z + M_z.$$

Then the tensor of the inertia is equal to

$$I = \begin{vmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{vmatrix}$$

here I_x, I_y, I_z are the principal moments of the inertia. The Euler's equations along the principal axes are expressed as

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= L_x + M_x \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= L_y + M_y \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= L_z + M_z. \end{aligned} \qquad (6,16)$$

Example:

Let us examine the rotation of the body having a variable mass about a fixed axis of the rotation. Choose the system of coordinates invariable associated with the body. Take oz as the axis of the rotation and let the other two axes ox and oy be orthogoral to it (Fig.10). Take the origin of the chosen coordinate system to be fixed. Suppose the inertial center of the body C (x_c, y_c, z_c)

does not change its position.

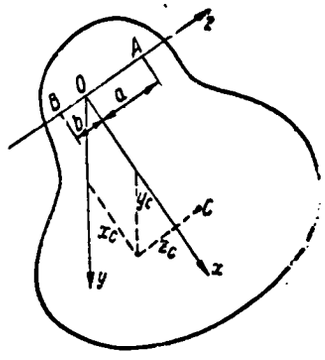

Рис. 10

Then

$$\omega_x = \omega_y = 0, \omega_z = \omega; \quad l_x = -I_{xz}\omega, \; l_y = -I_{yz}\omega,$$

$$l_z = I_z\omega.$$

Let the reactions have the projections

$$\bar{A} = (A_x, A_y, A_z), \quad \bar{B} = (B_x, B_y, B_z).$$

Since

$$\bar{v}_C = \bar{v}_C^{\bullet} = \bar{\omega} \times \bar{r}_C = (-\omega y_C, \omega x_C, 0), \; x_C = \text{const}, \; y_C = \text{const},$$

therefore, equations (6.10) can be written in the form

$$-\dot{\omega} y_C - \omega^2 x_C = \frac{1}{m}(F_{1x} + R_x + A_x + B_x),$$

$$\dot{\omega} x_C - \omega^2 y_C = \frac{1}{m}(F_{1y} + R_y + A_y + B_y),$$

$$0 = F_{1z} + R_z + A_z + B_z.$$

Here F_{1x}, F_{1y}, F_{1z} denote the projections of the principal vector of the active forces on x, y, z axes.

Equation (6.15) can be expressed as

$$- I_{xz}\dot{\omega} + I_{yz}\omega^2 = L_{1x} + M_x - a A_y + bB_y,$$

$$- I_{yz}\dot{\omega} - I_{xz}\omega^2 = I_{1y} + M_y + aA_x - bB_x,$$

$$I_z\dot{\omega} = L_{1z} + M_z.$$

Here L_{1x}, L_{1xy}, L_{1z} represent the moments of the active forces in respect of the axes of x,y,z.

The last equation of the above system enables us to determine the angular velocity of the rotation of the body by integration. The remaining two equations together with the three equations of the previous system and the additional conditions on A_z and B_z given law of change of the mass make it possible to the reactions in projections on the axes which are invariably associated with the body.

3. Euler's Equation of Rotation of the Solid Body having a Variable Mass about an Arbitrary Point A taken As a Pole.

The kinetic moment of the body for a fixed point is equal to

$$\bar{l} = \bar{l}_A + m(\bar{r}_C - \bar{r}_A) \times \bar{v} + \bar{r}_A \times \bar{Q}. \qquad (6.17)$$

Where $\bar{l}_A$ denotes the kinetic moment of the body for the point A in a relative motion in respect of the axes system, moving with the point, A. Consequently, $\bar{l}_A$ can be expressed as a volume integral over the body

$$\bar{l}_A = \iiint_{(\tau)} (\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \rho \, dxdydz, \qquad (6.18)$$

In (6.17), m represents the mass of the body, $\bar{r}_C$ - radius vector of its center of inertia, $\bar{r}_A$ - radius vector of the point A; $\bar{v}_A$ - velocity of the point A; Q - principal vector of the momentum of the body.

Uniting (1.8), (2.4) and (6.17), we obtain

$$\frac{d^*}{dt} \bar{l}_A + m (\bar{r}_C - \bar{r}_A) \times \bar{w}_A = \bar{L}_A + \bar{M}_A \qquad (6.19)$$

In this w_A denotes acceleration of the point A and $\bar{L}_A$ and $\bar{M}_A$ denotes the principal moments of the external and the reaction forces for the point A.

In view of the Euler's formula

$$\bar{v} - \bar{v}_A = \bar{\omega} \times (\bar{r} - \bar{r}_A)$$

where $\bar{\omega}$ is the angular velocity of the body, (6.18) can be written in the form

$$\bar{l}_A = I\bar{\omega} \qquad (6.20)$$

The inertial tensor

$$I = \begin{Vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{Vmatrix}$$

is computed in respect of the system of coordinate axes x,y,z with origin at the point A and firmly connected with the solid body.

Taking into consideration (6.14) and (6.20) the vector equation of (6.19) is obtained in the form

$$I \frac{d'\bar{\omega}}{dt} + \bar{\omega} \times I \bar{\omega} + m (\bar{r}_C - \bar{r}_A) \times \bar{w}_A = \bar{L}_A + \bar{M}_A. \qquad (6.21)$$

In view of the equation

$$\bar{w}_A = \frac{d}{dt} \bar{v}_A \frac{d'\bar{v}_A}{dt} + \bar{\omega} \times \bar{v}_A,$$

it becomes

$$I \frac{d'\omega}{dt} + m (\bar{r}_C - \bar{r}_A) \times \frac{d'\bar{v}_A}{dt} + m (\bar{r}_C - \bar{r}_A) \times (\bar{\omega} \times \bar{v}_A) + \qquad (6.22)$$

$$+ \bar{\omega} \times I \bar{\omega} = \bar{L}_A + \bar{M}_A$$

Since the projections of the velocity of the point A on the axes of x,y,z are α_A, β_A and γ_A, then the projections of the angular velocity, as before, are denoted by $\omega_x, \omega_y, \omega_z$; coordinates of the inertial center in this system are ξ, η and ζ. Projecting the vector equation (6.22) on the above referred axes x,y,z, we obtain Euler's three equations which determine the rotation of the solid body having a variable mass about an arbitrary point A of the body. These are

$$
\begin{aligned}
&I_x \dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z + m(\eta\dot{\gamma}_A - \zeta\dot{\beta}_A) = \\
&= m\left[\eta(\alpha_A\omega_y - \beta_A\omega_x) + \zeta(\alpha_A\omega_z - \gamma_A\omega_x)\right] + \\
&+ (I_y - I_z)\omega_y\omega_z + I_{xz}\omega_x\omega_y - I_{xy}\omega_x\omega_z + \\
&+ I_{yz}(\omega_y^2 - \omega_z^2) + L_{Ax} + M_{Ax}, \\
&- I_{xy}\dot{\omega}_x + I_y\dot{\omega}_y - I_{yz}\dot{\omega}_z + m(\zeta\alpha_A - \xi\dot{\gamma}_A) = \\
&= m\left[\zeta(\beta_A\omega_z - \gamma_A\omega_y) + \xi(\beta_A\omega_x - \alpha_A\omega_y)\right] + (I_z - I_x)\omega_x\omega_z + \\
&+ I_{xy}\omega_y\omega_z - I_{yz}\omega_x\omega_y + I_{xz}(\omega_z^2 - \omega_x^2) + L_{Ay} + M_{Ay}, \\
&- I_{xz}\dot{\omega}_x - I_{xz}\dot{\omega}_y + I_z\dot{\omega}_z + m(\quad_A - \quad_A) = \\
&= m\quad(\quad_A x - \quad_A z) + (\quad_A y - \quad_A z) + (I_x - I_y)\quad x\ y + \\
&+ I_{yz}\quad x\ z - I_{xz}\quad y\ z + I_{xy}(\quad_x^2 - \quad_y^2) + L_{Az} + M_{Az}
\end{aligned}
\tag{6.23}
$$

If the inertial center of the body having a variable mass remains fixed, then we may choose the inertial center itself as the pole A. In this case, vector, equation (6.22) becomes

$$I\,\frac{d'\bar{\omega}}{dt} + \bar{\omega} \times I\,\bar{\omega} = \bar{L}_C + \bar{M}_C \qquad (6.24)$$

and the Euler's equations for the rotation of the solid body will coincide with equations (6.15) in which L_x, L_y, L_z, M_x, M_y, M_z are replaced by L_{cx}, L_{cy}, L_{cz}, M_{cx}, M_{cy}, M_{cz}.

In case the inertial center changes its position equation (6.24) and consequently the Euler's equations will hold for the system of the axes unalterably connected with the body and having the origin at a point of the body, which at a given moment of time coincides with the center of the mass. Therefore, in such a study for each moment of time we must consider a particular set of axes.

4. Euler's Equations Using Known Angular Velocity of Rotation of the Principal Axes.

Let us determine the angular velocity $\bar{\Omega}$ of the rotation of the principal axes of inertia for the body. Consider the system of axes (x_1, y_1, z_1) which, at each moment of time, coincides with the principal directions and whose origin coincides with the center of the mass, which, in simple, is supposed not to move in a relation to the body.

Equation (6.19) thus becomes

$$\frac{d^*}{dt}\bar{l}_C = \bar{L}_C + \bar{M}_C, \qquad (6.25)$$

where l_C is represented in terms of the projections on the axes x_1, y_1, z_1 in the form

$$\bar{l}_C = I_1 \begin{Vmatrix} \omega_{x_1} \\ \omega_{y_1} \\ \omega_{z_1} \end{Vmatrix}, \quad I_1 = \begin{Vmatrix} I_{x_1} & 0 & 0 \\ 0 & I_{y_1} & 0 \\ 0 & 0 & I_{z_1} \end{Vmatrix} \qquad (6.26)$$

$\omega_{x_1}, \omega_{y_1}, \omega_{z_1}$ are the projections of the angular velocity of the rotation of the body on the principal axes, inertial moments about which are respectively equal to I_{x_1}, I_{y_1}, I_{z_1}.

Because the derivative $\frac{d^*}{dt} l_e$ is obtained while securing a mass of points, for calculating the left hand side of the formula (6.25) we must, therefore, consider the singular vectors of axis x_1, Y_1, z_1 in formula (6.26) as irreversable in relation to the body.

Consequently the derivative is computed in the system firmly connected with the body and which, at a given time, coincides with the system (x_1 y_1, z_1). We thus have

$$\frac{d^*}{dt}\bar{l}_C = \frac{d'^*}{dt}(I_1\bar{\omega}) + \bar{\omega} \times I_1\bar{\omega}. \qquad (6.27)$$

when the derivative with prime relates to the system firmly connected

with the body.

We now turn to the differentiation in the system (x_1, y_1, z_1) revolving together with theprincipal axes. According to the formula, we write the coupling (relative) derivatives in different systems

$$\frac{d''}{dt}(I_1\bar{\omega}) = I_1 \left\| \begin{matrix} \dot{\omega}_{x_1} \\ \omega_{y_1} \\ \omega_{z_1} \end{matrix} \right\| + \bar{\Omega} \times I_1\omega \qquad (6.28)$$

Projecting equation (6.25) on the axes x,y,z, with the help of equations (6.24) and (6.28) we get

$$\begin{aligned}
&I_x\dot{\omega}_{x_1} + (I_{z_1} - I_{y_1})\omega_{y_1}\omega_{z_1} + I_{z_1}\Omega_{y_1}\omega_{z_1} - I_{y_1}\Omega_{z_1}\omega_{y_1} = L_{x_1} + M_{x_1} \\
&I_{y_1}\dot{\omega}_{y_1} + (I_{x_1} - I_{z_1})\omega_{x_1}\omega_{z_1} + I_{x_1}\Omega_{z_1}\omega_{x_1} - I_{z_1}\Omega_{x_1}\omega_{z_1} = L_{y_1} + M_{y_1} \qquad (6.29) \\
&I_{z_1}\dot{\omega}_{z_1} + (I_{y_1} - I_{x_1})\dot{\omega}_{x_1}\omega_{y_1} + I_{y_1}\Omega_{x_1}\omega_{y_1} - I_{x_1}\Omega_{y_1}\omega_{x_1} = L_{z_1} + M_{z_1}
\end{aligned}$$

Formula (6.29) furnishes the required Euler's equations. The explicit expressions for $\Omega_{x_1}, \Omega_{y_1}, \Omega_{z_1}$ will be considered in section 4 of the Chapter V.

5. Meshersky Case:

Suppose there does not occur a relative motion of particles in the body having a variable mass. Then for all the particles

$$\bar{v}_r = 0, \quad \bar{w}_r = 0, \quad \bar{w}_C = 0.$$

Consequently, the principal vector and the principal moment of reaction forces about any pole will be equal to the principal vector and the principal moment of the impulsive forces

$$\bar{R} = \bar{\Phi}, \quad \bar{M} = \bar{K}$$

In this process, change of the mass takes place only in the points on the surface of the body. Therefore, we have

$$\bar{\Phi} = \iint_{(S)} (\bar{u} - \bar{v}) \frac{\partial \nu}{\partial t} ds, \quad \bar{K} = \iint_{(S)} [\bar{r} \times (\bar{u} - \bar{v}) \frac{\partial \nu}{\partial t} ds.$$

Here S represents the surface enclosing the body, ds - surface element, ν (x, y, z) - density of the surface layer, whose mass varies.

In the context of the Meshersky case concerning the solid body having a variable mass, the laws of change of the principal vector of the momentum and kinetic moment, and then also the equation of motion of the inertial center and the Euler's equations were worked out by A.A.Kosmodemyanski in 1946 [33,36,35]. The laws

of motion of the body having a variable mass have been nicely treated in a course of theoretical mechanics written by L.G. Luitsyanskiĭ and A.I. Lurc 45.

6. Principal of Solidification:

Summing up the results so far achieved it can be stated that the laws of change of the principal vector of the momentum and the kinetic moment for the body with a variable mass and also the fundamental equations derived from them have some form as in the case of a body with constant mass, measured for an excessive solidification if we add the principal vector and the principal moment of reaction forces to the external forces and the moments. This principle may be called the principle of solidification for constructing the equation of motion. This principle first appeared in the works of F.R. Mantmakhera, L.L. Levina (11,12) and I.G. Luitsyantsko, A.I. Lurc'.

7. Laws of Change of the Kinetic Energy in the System of Points of a Variable Mass:

1. Change of the Kinetic Energy and the Additional Field of Forces in the Mechanics of a Variable Mass:

The equation of motion of points of a variable mass in case of a relative motion of particles wasobtained in Chapter I in the form

$$m \frac{d\bar{v}}{dt} = \bar{F} + \bar{R}.$$

The reaction force $\bar{R}$ is determined from the formula

$$\bar{R} = \bar{J} - \sum m\bar{w}_r + \bar{\Phi}$$

where $\bar{J}$ is Cariliso force and denotes the impulsive force. $\bar{\Phi}$ The summation in the middle part on the right side is carried over all the particles building the material point at time t; $\bar{w}_r$ - denotes the relative accelerations of the particles whose masses are denoted by m. If there does not take place a relative motion of particles in the material point, then the motion of the material point having a variable mass is determined from the Meshersky equation

$$m\frac{d\bar{v}}{dt} = \bar{F} + \frac{dm_1}{dt}(\bar{u}_1 - \bar{v}) - \frac{dm_2}{dt}(\bar{u}_2 - \bar{v}).$$

It may be observed that the equation of motion material point in theory of relativity, viz.,

$$\frac{d}{dt}\frac{m_o\bar{v}}{\sqrt{1-\varphi^2}} = \bar{F}, \quad \beta = \frac{v}{c} \tag{7.1}$$

where c represents the velocity of light, may be considered a particular case of the Meshersky equation if we take

$$m = \frac{m_o}{\sqrt{1-\beta^2}} \tag{7.2}$$

and the absolute velocities of the changing masses equal to zero. In this case we will have

$$\bar{R} = -\bar{v}\frac{d}{dt}\frac{m_o}{\sqrt{1-\beta^2}} \tag{7.3}$$

Although the above analogy is formal, however, in the mechanics of the theory of relativity we come across Langranges and the Hamiltonian function as well as the motion of the kinetic energy. Therefore, the mechanics of variable masses is naturally orientated towards the corresponding concepts of the mechanics of the theory of relativity.

Let us extend the motion of the kinetic energy into the mechanics of a variable mass. The mechanics of a constant mass employs in place of the kinetic energy such a physical quantity where the change gives the work done by the force applied to the specified material point. A similar principle works in the theory of relativity. The scalar multiplication of the vectorial equation (7.1) by the velocity v_1 gives

$$d \frac{m_o c^2}{\sqrt{1 - \beta^2}} = \bar{F} d\bar{r} \qquad (7.4)$$

Thus in the mechanics of the theory of relativity the kinetic energy is calculated to be

$$T = (m - m_o) c^2$$

Here mass m is determined from (7.2) and consequently T = 0 when v = 0.

But it is futile to try to come close to such a concept of the kinetic energy in the mechanics of a variable mass.

As a matter of fact, scalare multiplying the equation of the

motion of a point of variable mass by the velocity v, we find

$$m\bar{v}d\bar{v} - \bar{R}d\bar{r} = \bar{F}d\bar{r} \qquad (7.5)$$

It appears that the left hand side of (7.5) is the total differential. Considering F = 0, suppose that the process of change takes place with certain velocities, we will have = constant which entails a physical hitch. It is not convenient to consider the kinetic energy as a physical quantity which remains unchanged with the increase or decrease of energy. In the mechanics of theory of relativity the above referred contradiction does not exit because when F = 0, we get $\bar{v}$ = constant, $\frac{m_o}{\sqrt{1-\beta^2}}$ = const.

We obtain the kinetic energy of a point of a variable mass in the form

$$T = \frac{mv^2}{2}$$

so that it corresponds to the kinetic energy of the point when the process of change of mass completely stopped beginning from a given moment.

Relation (7.5) becomes

$$d^*T = \bar{F}d\bar{r} + \bar{R}d\bar{r} \qquad (7.6)$$

Where the scalar product Rdr represents the elementary work done by the reaction force. On the left of equation (7.6) stands partial

differential with m and t as independent variables and m fixed.

Expression (7.6) may be put in the form

$$d\bar{T} = \bar{F}d\bar{r} + \bar{D}t\bar{r}, \qquad (7.7)$$

where

$$\bar{D} = \bar{R} + \frac{dm}{dt}\,\frac{\bar{v}}{2} \qquad (7.8)$$

is called the additional force or the force of additional field. $D\,d\bar{r}$ is called the elementary work of this field.

In the theory of relativity $\bar{D} = -\frac{\bar{v}}{2}\frac{dm}{dt}$, and the additional field turns out to the potential because when the mass changes by (7.2), we have

$$-\frac{v^2}{2}\,dm = d\left(\frac{mv^2}{2} - mc^2\right) \qquad (7.9)$$

For a system of points having a variable mass we get

$$T = \sum_j \frac{m_j v_j^2}{2}$$

$$d^*T = \sum_j \bar{F}_j d\bar{r}_j + \sum_j \bar{R}_j d\bar{r}_j \qquad (7.10)$$

(7.10) shows that the increment in the kinetic energy of the solidified system is equal to the sum of elementary works done by

the usual and the reaction forces. Suppose the field of the usual forces in case of constant masses possesses potential energy II, then

$$d^* (T - \Pi) = \sum_j R_j d\bar{r}_j \qquad (7.11)$$

Energy equation (7.10) can be expressed in the form

$$dT = \sum_j \bar{F}_j d\bar{r}_j + \sum_j \bar{D}_j d\bar{r}_j \qquad (7.12)$$

where

$$\sum_j \bar{D}_j d\bar{r}_j = \sum_j (\bar{R}_j + \frac{dm_j}{dt} \frac{\bar{v}_j}{2}) d\bar{r}_j \qquad (7.13)$$

is the elementary work done by the force of additional field.

Integrating (7.12) along the trajectory of motion of the system we get

$$T = T_o = \int \sum_j \bar{F}_j d\bar{r}_j + \int \sum_j \bar{D}_j d\bar{r}_j \qquad (7.14)$$

Thus the increment in the kinetic energy of the system of points having a variable mass is equal to the work done by the usual forces of additional field.

It is important to note that the work done by the usual forces splits into work done by the internal forces so that in the case of

a solid body having a variable mass the latter work reduces to zero.

2. Conditions of the Potentiality and Quasi-Potentiality of the Additional Field of Forces.

The position of the mechanical system is completely determined by S generalized Lagrangian coordinates q_i (i = 1, 2, ..., S). The masses of the points are functions of times, langrangian coordinates and the velocities

$$m_j = m_j (q_i, \dot{q}_i, t) \qquad (7.15)$$

We denote the elementary work done by the forces of additional field in terms of generalized coordinates. (7.15) shows that the reaction forces are linearly dependent upon Langrangian accelerations

$$\bar{R}_j = \sum_{k=1}^{S} \bar{R}_{jk} (q_i, \dot{q}_i, t)\, \ddot{q}_k + \bar{R}_{j0} (q_i, \dot{q}_i, t) \qquad (7.16)$$

Therefore we get

$$\sum_j \bar{D}_j d\bar{r}_j = \sum_{i=1}^{S} \sum_j \left(\bar{R}_{j0} \frac{\partial \bar{r}_j}{\partial q_i} + \frac{v_j^2}{2} \frac{\partial m_j}{\partial q_i}\right) dq_i +$$

$$+ \sum_{i=1}^{S} \sum_j \left(\bar{R}_{ji} \bar{v}_j + \frac{v_j^2}{2} \frac{\partial m_j}{\partial q_i}\right) d\dot{q}_i + \sum_j \left(\bar{R}_{j0} \frac{\partial \bar{r}_j}{\partial t} + \frac{v_j^2}{2} \frac{\partial m_j}{\partial t}\right) dt$$

....(7.17)

If (7.17) works out to be the total differential, then the additional field has potential. For holonomous system this is a necessary and sufficient condition.

The subject matter of section 7 has been dealt with by the author in his work [58].

8. GENERAL EQUATION OF THE MECHANICS OF A VARIABLE MASS:

1. General Equation in Cartesian Coordinates:

As seen in section 5, in case of a solid body we choose geometrical points of the body as material points of variable masses, i.e., small material regions moving translationally with the chosen geometrical points of the fixed system. Thus the mechanics of the system with variable masses completely preserves the kinematics of points and solid bodies having a constant mass.

The motion of the system of points a with variable mass may experience certain holonomic and non-holonomic constraints. The constraints are assumed to be ideal.

From equation of the motion of point (1.2) in Chapter I and the condition of ideal constraint, the general equation of the mechanics of a variable mass follows

$$\sum_j \left(m_j \frac{d\bar{v}_j}{dt} - \bar{F}_j - \bar{R}_j\right) \delta \bar{r}_j = 0 \qquad (8.1)$$

Where $\bar{F}_j$ is the resultant active force applied on jth-point.

Equation (8.1) has direct applications inthe concrete problems and is also used in the derivation of different equations of motion in generalized coordinates.

Examples of the direct application are given in the works [58] of the author.

2. General Equation in the Langrangian Coordinates:

Suppose the masses of the points of the mechanical system are expressed by relations of the type (7.15). Then the general equation (8.1) of the mechanics of a variable mass can be written in generalized coordinates.

$$\sum_{i-1}^{S} \left(\frac{d}{dt} \frac{\partial T}{\partial q_i} - \frac{\partial T}{\partial q_i} - Q_i - P_i \right) \delta q_i = 0 \qquad (8.2)$$

Here

$$Q_i = \sum_j \bar{F}_j \frac{\partial \bar{r}_j}{\partial q_i}$$

are generalized active forces,

$$P_i = \sum_j \left[\left(\bar{D}_j + \frac{dm_j}{dt} \frac{\bar{v}_j}{2} \right) \frac{\partial \bar{r}_j}{\partial q_i} - \frac{v_j^2}{2} \frac{\partial m_j}{\partial q_i} + \frac{d}{dt} \left(\frac{v_j^2}{2} \frac{\partial m_j}{\partial q_i} \right) \right] \qquad (8.3)$$

is termed as surplus forces

Denote by

$$\frac{\partial^* T}{\partial q_i}, \frac{\partial^* T}{\partial \dot{q}_i} \quad (i = 1, 2, \ldots, S)$$

the partial derivatives for the indicated variables in the set of independent variables

$$m_j,\ t,\ q_i,\ \dot{q}_i,$$

Taking into consideration the equation

$$\frac{d\bar{v}_i}{dt}\frac{\partial \bar{r}_i}{\partial q_i} = \frac{d}{dt}\frac{\partial \frac{v_i^2}{2}}{\partial \dot{q}_i} - \frac{\partial \frac{v_i^2}{2}}{\partial q_i}$$

The general equation (8.1) can be written in the form

$$\sum_{i=1}^{S} \left(\frac{d^*}{dt}\frac{d^*T}{\partial \dot{q}_i} - \frac{\partial^* T}{\partial q_i} - Q_i - \Psi_i \right) \delta q_i = 0 \qquad (8.4)$$

$$\Psi_i = \sum_{r} \bar{R}_j \frac{\partial \bar{r}_i}{\partial q_i} \quad (i = 1, 2, \ldots, S) \qquad (8.5)$$

represents the generalized active forces.

In deriving the equation (8.4) no restrictions were imposed on the functional dependence of masses, therefore, the masses system under examination besides generalized coordinates, its derivatives and the time may also depend upon the distances S_r traversed by rth point.

$$m_j = m_j (q_i, \dot{q}_i, S_r, t) \quad (i=1, 2, \ldots, S;\ r = 1, 2, \ldots, n) \qquad (8.6)$$

Therefore, equation (8.4) may be adopted under lesser limitations than the equation (8.2)

CHAPTER - III

HOLONONIC SYSTEMS WITH VARIABLE MASSES.

7. Lagrange's Equations of the Second Type.

1. Two forms of Lagranges equations of the second type in the mechanics of a variable mass. Let us examine holonomic systems. For this purpose, choose the generalized coordinates q_i ($i = 1, 2, \ldots, S$) which satisfy all the constraints imposed on the motion of a mechanical system. This condition will make the coordinates q_i independent and hence the variation δq_i in them will also be independent.

Taking into account the independence of variations q_i and the general equation of the mechanics of variable mass (8.2) in chapter II, we obtain S equations in generalized coordinates

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + P_i \quad (i = 1, 2, \ldots, S). \tag{1.1}$$

The surplus forces are determined by the formula (8.3) in chapter II. In the case of a body having a variable mass sum of the terms for the values of the range j is equivalent to the integral over the volume of the body

$$P_i = \int\int\int_{(\tau)} \left[\left(\bar{R} + \frac{d\rho}{dt}\bar{v}\right)\frac{\partial \bar{r}}{\partial q_i} - \frac{v^2}{2}\frac{\partial \rho}{\partial q_i} + \frac{d}{dt}\left(\frac{v^2}{2}\frac{\partial \rho}{\partial \dot{q}_i}\right)\right] dxdydz. \tag{1.2}$$

In this, $\bar{R}$ is reaction force; $\rho = \rho\ (q_i, \dot{q}_i, t; x, y, z)$ - is dnsity at the point (x, y, z) of the body.

Equation (1.1) are called Lagranges generalized equations of the second type. We will discuss particular cases of the equations (1.1).

The masses of the points will be explicit functions of time only, i.e. $m_j = m_j(t)$, uniting relations (7.8) and (8.3) of chapter II, we have

$$P_i = \sum_j \left(\bar{R}_j + \frac{dm_j}{dt}\bar{v}_j\right)\frac{\partial \bar{r}_j}{\partial q_i}. \qquad (1.3)$$

In the Meshcherkys case, i.e., when there is no relative motion of the particles in the material points of the mechanical system reaction force of j-th point is equal to

$$\bar{R}_j = \frac{dm_{1j}}{dt}(\bar{u}_{1j} - \bar{v}_j) - \frac{dm_{2j}}{dt}(\bar{u}_{2j} - \bar{v}_j). \qquad (1.4)$$

Therefore, in this case,

$$P_i = \sum_j \left(\frac{dm_{1j}}{dt}u_{ij} - \frac{dm_{2j}}{dt}\bar{u}_{2j}\right)\frac{\partial \bar{r}_j}{\partial q_i}. \qquad (1.5)$$

Lagrange's equations of the type (1.1) were obtained by in his work [36] in the year 1951.

Let the absolute velocities of the changing masses be equal to zero so that $\bar{u}_2 = 0$, then $P_i = 0$ $(i = 1, 2, \ldots, s)$ and equations (1.1) take the same form as in the mechanics of constant masses. This was shown by I.V. Meshcherky in his thesis (53) in the year 1897.

Taking into account the general equation in the mechanics of variable mass (8.4) in chapter II we obtain the Lagranges equation of the second type in the form

$$\frac{d^*}{dt}\frac{\partial^* T}{\partial \dot{q}_i} - \frac{\partial^* T}{\partial q_i} = Q_i + \Psi_i \quad (i = 1, 2, \ldots, s). \qquad (1.6)$$

Equation (1.1) may be immediately transformed into the form (1.6), but then it must be assumed that the mass functions (7.5) in chapter II are dependent.

Independent derivation of equations (1.6) enables us to extend the use of these equations to any functional dependence relation of a mass of points. In particular, these equations can be conveniently used for a dependence relation of the type (8.6) in chpater. Add to the equations (1.6) the differential equations

$$\frac{dS_r}{dt} = \sqrt{\frac{2T_r}{m_r}} \quad (r = 1, 2, \ldots, n), \qquad (1.7)$$

where the kinetic energy of r-th point T_r and its mass m_r must be expressed in terms of generalized coordinates, their derivatives, time and the distances S_r travelled by r points of the system.

The generalized action forces for consolidated masses figure out to be potentials $Q_i = -\frac{\partial \Pi}{\partial q_i}$, and taking into account the Lagranges function $L = T - \Pi$ equation (1.6) may be written in the form

$$\frac{d^*}{dt}\frac{\partial^* L}{\partial \dot{q}_i} - \frac{\partial^* L}{\partial q_i} = \Psi_i. \qquad (1.8)$$

Derivatives on the left hand side of equations (1.6) and (1.8) are obtained independent of mass, hence, for compiling equation (1.6) and (1.8) it is necessary to use Lagrange's equations for the corresponding problem with constant masses and then to obtain masses and add generalized reactive forces. The above rule can be considered a generalization of the solidification principle to be added to Lagraange's

equations. The left portions of Lagrange's equations(1.6) and (1.8) represent the left portion of Lagrange's equations in the solidified system.

Considering dependence of a mass on velocities we must assume the linear dependence of the generalized reaction forces on accelerations.

$$\Psi_k = \sum_{\gamma=1}^{s} \Psi_{k\gamma}(q_i, \dot{q}_i, t)\,\ddot{q}_\gamma + \Psi_{ko}(q_i, \dot{q}_i, t) \quad (k, i = 1, 2, \ldots, s). \tag{1.9}$$

Therefore, Langranges equations (1.6) in respect of acceleration will be solved only when certain conditions are satisifed. Actually, the kinetic energy is equal to $T = T_2 + T_1 + T_o$, where T_p $(p = 0, 1, 2)$ is a suitable homogeneous form in generalized velocities, for in such an investigation the masses are consolidated. The quadratic part of the kinetic energy is written in the form

$$T_2 = \sum_{k,\gamma=1}^{s} a_{k\gamma}(m_j, q_i, t)\,\dot{q}_k\dot{q}_\gamma, \quad a_{k\gamma} = a_{\gamma k},$$

then, the condition for the existence of the solution is that the determinant of the matrix $\|a_{k_\gamma} - \Psi_{k\gamma}\|$ is different from zero.

2. Example: Two uniform circular cylinders A and B are tied with two flexible threads. The weight of each cylinder is G and the radius of the base is r. The tending are symmetrically inclined to the medium (central) planes parallel to the bases of the cylinder. The axes of the cylinders are horizontal (fig. 11). The axis of the cylender B drops from the rest under the action of the gravity and stretches the two flexible fastenings
The other ends of the claims are flung over infinitesimally small blocks and these together with considerable part of chais rest on a rough

horizontal table. The friction coefficient of the chain is k.

Determine the angular velocities $\dot{\varphi}_1$ amd $\dot{\varphi}_2$ as functions of the displacement S moved by the cylender B.

The axis of the cylinder B will descend vertically and, therefore, it may be thought that the system possesses two degree freedom. The angles φ_1 and φ_2 may be taken as generalized coordinates. Let $\varphi_{10} = \varphi_{20} = 0$.

The kinetic energy of the system is equal to

$$T = \frac{Gr^2}{4g}(\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + \frac{G}{2g}S^2 + \gamma(1 + S)\dot{S}^2,$$

where S is the displacement of the axis of the cylinder B; $S = (\varphi_1 + \varphi_2)r$. The virtual work done by the reaction forces is given by the expression

$$\left[Gr + 2g\gamma(1 - b + S)\,r - 2g\gamma kbr\right](\delta\varphi_1 + \delta\varphi_2).$$

Virtual work done by reaction forces is given by the expression

$$-2\gamma r^3(\dot{\varphi}_1 + \dot{\varphi}_2)^2(\delta\varphi_1 + \delta\varphi_2).$$

In the present case, Lagranges equations (1.6) are written in the form

$$\left[\frac{3}{2}\frac{G}{g} + 2\gamma(1 + S)\right] r\ddot{\varphi}_1 + \left[\frac{G}{g} + 2\gamma(1 + S)\right] r\ddot{\varphi}_2 =$$

$$= G + 2g\,(1 - b + S) - 2g\gamma kb - 2\gamma r^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2,$$

$$\left[\frac{G}{g} + 2\gamma(1 + S)\right] r\ddot{\varphi}_1 + \left[\frac{3}{2}\ \frac{G}{g} + 2\gamma(1 + S)\right] r\ddot{\varphi}_2 =$$

$$= G + 2g\gamma(1 - b + S) - 2g\gamma kb - 2\gamma r^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2.$$

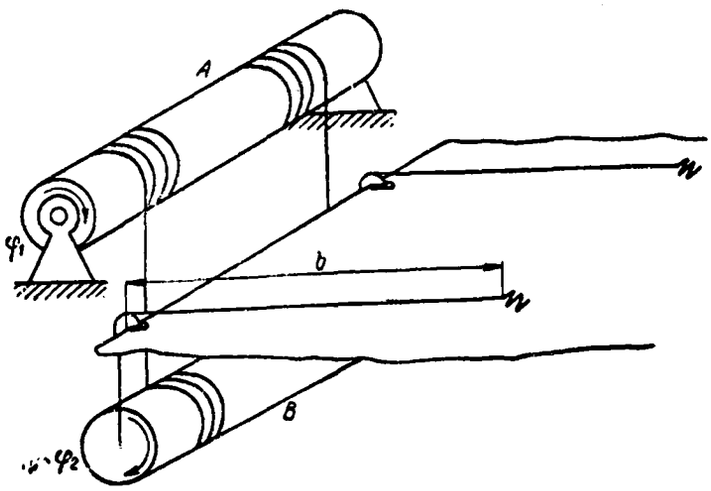

Рис. 11

Figure - 11

From here we must conclude that

$$\dot{\varphi}_1 = \dot{\varphi}_2 = \frac{1}{2}\ \frac{S}{r}\ ,\ \left[\frac{5}{4}\ \frac{G}{g} + 2\gamma(1 + 2r\varphi_1)\right] r\ \frac{d\dot{\varphi}_1^2}{d\varphi_1} +$$

$$+ 8gr^2\dot{\varphi}_1^2 = G + 2g\gamma(1 - b + 2r\varphi_1) - 2g\gamma kb.$$

Thus a linear differential equation of the first order in $\dot{\varphi}_1^2$ wrt φ_1, is obtained from which $\dot{\varphi}_1^2$ is determined as a function of φ_1 or as a function of displacements S.

In the work (58) of the author, another example is discussed to demonstrate the application of the Lagranges equation of the second type.

2. Lagrange's Equations Together with Lagrange's Function and Cannonical Equations in the Mechanics of Variable Mass.

1.

Conditions for the existence of the generalized kinetic potential of the surplus forces. Generalized surplus forces P_i are linearly dependent upon the acceleration, and therefore, we apply on them conditions (101, 111, 113) for the existence of the generalized kinetic potential.

$$\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i} = 0 \quad (i, k = 1, 2, \dots, s), \tag{2.1}$$

$$\frac{\partial P_i}{\partial \dot{q}_k} + \frac{\partial P_k}{\partial \dot{q}_i} = 2 \frac{d}{dt} \frac{\partial P_k}{\partial q_i}, \tag{2.2}$$

$$\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i} = \frac{1}{2} \frac{d}{dt} \left(\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i} \right). \tag{2.3}$$

When conditions (2.1) - (2.3) are satisfied, the surplus forces are given by

$$P_i = \frac{\partial \Theta}{\partial q_i} - \frac{d}{dt} \frac{\partial \Theta}{\partial \eta_i} \tag{2.4}$$

where Θ denotes a function of q_k, $\dot{q}_k$, t.

We will show a method for the computation of the explicit from the function Θ when* Helmholtz conditions are satisfied. The method will be fairly similar to that explained** Mayer (115).

We will find the function which satisfy the relations

$$\frac{\partial q_i}{\partial \dot{q}_k} = \frac{\partial P_k}{\partial \dot{q}_i} \quad (i, k = 1, 2, \ldots, s). \tag{2.5}$$

Applying condition (2.1), we have

$$\frac{\partial q_i}{\partial \dot{q}_k} = \frac{\partial q_k}{\partial \dot{q}_i} . \tag{2.6}$$

We will prove that the equation

$$\sum_{k=1}^{s} \frac{\partial P_i}{\partial \dot{q}_k} d\dot{q}_k = \sum_{k=1}^{s} \frac{\partial q_i}{\partial \dot{q}_k} d\dot{q}_k = d\chi_i . \tag{2.7}$$

holds true

Here $d\chi_i$ is total differential with q_k and t, as fixed. From the condition (2.2) we find

$$\frac{\partial}{\partial \dot{q}} \left(\frac{\partial P_k}{\partial \dot{q}_i} + \frac{\partial P_i}{\partial \dot{q}_k}\right) = 2 \frac{\partial^2 P_i}{\partial \dot{q}_\gamma \partial q_k} ,$$

$$\frac{\partial}{\partial \dot{q}_k} \left(\frac{\partial P_\gamma}{\partial \dot{q}_i} + \frac{\partial P_i}{\partial \dot{q}_\gamma}\right) = 2 \frac{\partial^2 P_i}{\partial \dot{q}_k \partial q_\gamma} .$$

subtracting the second form the first we get

$$\frac{\partial}{\partial \dot{q}_i} \left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial \dot{q}_k}\right) = 3 \left(\frac{\partial^2 P_i}{\partial \dot{q}_\gamma \partial \dot{q}_k} - \frac{\partial^2 P_i}{\partial \dot{q}_k \partial \dot{q}_\gamma}\right). \tag{2.8}$$

Applying (2.1) to the left hand side it becomes zero, and we obtain the equation

$$\frac{\partial^2 P_i}{\partial \dot{q}_\gamma \partial q_k} = \frac{\partial^2 P_i}{\partial \dot{q}_k \partial \dot{q}_\gamma} \quad (i, k, \gamma = 1, 2, \ldots, s). \tag{2.9}$$

Applying (2.7) we find the required functions φ_i in the form

$$\varphi_i = \chi_i (\dot{q}_k, q_k, t) + \omega_i (q_k, t)/ \qquad (2.10)$$

where ω_i are functions in variables q_k and t.

Denote

$$\Omega_{ik} = \frac{1}{2} \left(\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i}\right) - \left(\frac{\partial \chi_i}{\partial q_k} - \frac{\partial \chi_k}{\partial q_i}\right), \qquad (2.11)$$

$\Omega_{ik} = -\Omega_{ki}$

It will be shown that the functions Ω_{ik} depends upon Lagrangian coordinates and time only. Differentiating (2.11) wrt Lagrangian coordinates, we get

$$\frac{\partial}{\partial q_\gamma} \Omega_{ik} = \frac{1}{2} \frac{\partial}{\partial q_\gamma} \left(\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i}\right). \qquad (2.12)$$

Because of condition (2.3) right hand side is zero for, in the opposite case, left hand side of (2.3) must depend $\dot{q}_\gamma$. P_k are functions of the Lagrangian velocities, coordinates and time only and linearly depend upon the second order derivatives of the Lagrangian coordinates.

Differentiating (2.11) in respect of Lagrangian velocities, we have

$$\frac{\partial}{\partial \dot{q}_\gamma} \Omega_{ik} = \frac{1}{2} \frac{\partial}{\partial \dot{q}_\gamma} \left(\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial \dot{q}_\gamma} \left(\frac{\partial \chi_i}{\partial q_k} \frac{\partial \chi_k}{\partial q_i}\right). \qquad (2.13)$$

From condition (2.3) we get

$$\frac{1}{2} \frac{\partial}{\partial \dot{q}_\gamma} \left(\frac{\partial P_i}{\partial q_x} - \frac{\partial P_k}{\partial \dot{q}_i}\right) = \frac{\partial}{\partial \dot{q}_\gamma} \left(\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i}\right). \qquad (2.14)$$

Combining (2.13), (2.14) and (2.10) gives

$$\frac{\partial}{\partial \dot{q}_j}\Omega_{ik} = \frac{\partial}{\partial q_j}\left(\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i}\right) - \left(\frac{\partial^2 \varphi_i}{\partial q_k \partial \dot{q}_j} - \frac{\partial^2 \varphi_k}{\partial q_i \partial \dot{q}_j}\right).$$

Taking into consideration (2.5) and (2.6), we find

$$\frac{\partial^2 \varphi_i}{\partial q_k \partial \dot{q}_j} - \frac{\partial^2 \varphi_k}{\partial q_i \partial \dot{q}_j} = \frac{\partial^2 P_i}{\partial q_k \partial q_j} - \frac{\partial^2 P_k}{\partial q_i \partial q_j}.$$

From here evidently follows

$$\frac{\partial}{\partial \dot{q}_j}\Omega_{ik} = 0 \quad (i, k, j = 1, 2, \ldots, s).$$

Formulae (2.5) indicate the character of the functions φ_i with regards to its dependence on the Lagrangian velocities and its dependence on the Lagrangian coordinates has not yet been explained.

$$\frac{\partial \varphi_i}{\partial q_k} - \frac{\partial \varphi_k}{\partial q_i} = \frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{q}_k} - \frac{\partial P_k}{\partial \dot{q}_i}\right). \qquad (2.15)$$

Equation (2.15) together with (2.11) are equivalent to the following equations in terms of the unknown functions ω_i:

$$\frac{\partial \omega_i}{\partial q_k} - \frac{\partial \omega_k}{\partial q_i} = \Omega_{ik} \quad (i, k = 1, 2, \ldots, s). \qquad (2.16)$$

Equations (2.16) can be written in the following tabular form

$$\left|\begin{array}{l} \frac{\partial \omega_1}{\partial q_2} = \frac{\partial \omega_2}{\partial q_1} + \Omega_{12}, \quad \frac{\partial \omega_1}{\partial q_3} = \frac{\partial \omega_3}{\partial q_1} + \Omega_{13}, \quad \ldots \quad \frac{\partial \omega_1}{\partial q_s} = \frac{\partial \omega_s}{\partial q_1} + \Omega_{1s}, \\ \\ \qquad\qquad \frac{\partial \omega_2}{\partial q_3} = \frac{\partial \omega_3}{\partial q_2} + \Omega_{23}, \quad \ldots, \quad \frac{\partial \omega_2}{\partial q_s} = \frac{\partial \omega_s}{\partial q_2} + \Omega_{2s}, \\ \\ \qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right.$$

$$\frac{\partial \omega_{s-1}}{\partial q_s} = \frac{\partial \omega_s}{\partial q_{s-1}} + \Omega_{s-1,s}.$$

Function ω_s is chosen arbitrarily; from the last equation above find ω_{s-1} and thus proceeding all the remaining functions will be determined if the following equations are satisfied

$$\frac{\partial}{\partial q_\gamma}\left(\frac{\partial \omega_i}{\partial q_k} + \Omega_{ki}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial \omega_\gamma}{\partial q_k} + \Omega_{k\gamma}\right) \quad (1 \leq k < i < \gamma \leq s). \tag{2.17}$$

It is easy to see that conditions (2.17) are fulfilled. For values of i and γ and large k we will suppose that equations (2.16) are satisfied and hence the conditions (2.17) can be written as

$$\frac{\partial}{\partial q_i}\Omega_{k\gamma} + \frac{\partial}{\partial q_k}\Omega_{\gamma i} + \frac{\partial}{\partial q_\gamma}\Omega_{ik} = 0. \tag{2.18}$$

It will now be shown that relation (2.18) holds true. Put the values of Ω_{ik} from (2.11)

$$\frac{\partial}{\partial q_i}\left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial q_k}\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial P_\gamma}{\partial q_i} - \frac{\partial P_i}{\partial q_\gamma}\right) + \frac{\partial}{\partial q_\gamma}\left(\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i}\right) = 0. \tag{2.19}$$

Left hand side of (2.19) can also be written in the form

$$\frac{\partial}{\partial q_i}\left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial q_k}\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial P_\gamma}{\partial q_i} - \frac{\partial P_i}{\partial q_\gamma}\right) + \frac{\partial}{\partial q_\gamma}\left(\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i}\right). \tag{2.20}$$

Applying condition (2.3) we have

$$\frac{\partial}{\partial q_i}\left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial q_k}\right) = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial q_k}\right) + \frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial q_i}\left(\frac{\partial P_k}{\partial q_\gamma} - \frac{\partial P_\gamma}{\partial q_k}\right).$$

Therefore, left hand side of (2.19) is equal to double expression (2.20). From here follows the validity of (2.19) and, therefore, of the identity (2.18).

Now we will look for a function Θ which satisfied the conditions

$$\frac{\partial \Theta}{\partial q_i} = -\varphi_i, \quad \frac{\partial \Theta}{\partial q_i} = -\frac{d}{dt}\varphi_i + P_i.$$

If it is possible to find such a function, then it will be indentified and characterized as potential function for values P_i

We will prove the equality of the corresponding cross derivatives. Partial derivatives $\frac{\partial \varphi_i}{\partial q_k}$ and $\frac{\partial \varphi_k}{\partial q_i}$ are equal because of (2.6). Further we have

$$\frac{\partial}{\partial q_k}\left(-\frac{d}{dt}\varphi_i + P_i\right) - \frac{\partial}{\partial q_i}\left(-\frac{d}{dt}\varphi_k + P_k\right) =$$

$$= -\frac{d}{dt}\left(\frac{\partial \varphi_i}{\partial q_k} - \frac{\partial \varphi_k}{\partial q_i}\right) + \frac{\frac{\partial P_i}{\partial q_k} - P_k}{q_i}. \qquad (2.21)$$

As a result of (2.3) and (2.15) right hand side of (2.21) is equal to zero.

It remains to show that the relation

$$\frac{\partial}{\partial q_k}\left(-\varphi_i\right) = \frac{\partial}{\partial q_i}\left(-\frac{d}{dt}\varphi_k + P_k\right). \qquad (2.22)$$

We have

$$\frac{\partial}{\partial q_i}\frac{d}{dt}\varphi_k = \frac{d}{dt}\frac{\partial \varphi_k}{\partial q_i} + \frac{\partial \varphi_k}{\partial q_i} \qquad (2.23)$$

By (2.15) and (2.23) relations (2.22) are equivalent to the equalities

$$\frac{1}{2}\left(\frac{\partial P_i}{\partial \dot{q}_k} + \frac{\partial P_k}{\partial \dot{q}_i}\right) = \frac{a}{dt}\frac{\partial \varphi_k}{\partial \dot{q}_i}. \qquad (2.24)$$

Relations (2.24) hold true because of (2.2) and (2.5).

The function Θ is obtained as curvilinear integral which is independent of the path of integration

$$\Theta = \int_{(q_i^o, \dot{q}_i^o, t)}^{(q_i, \dot{q}_i, t)} \sum_{k=1}^{s} (-\varphi_k)\, d\dot{q}_k + \sum_{k=1}^{s} \left(-\frac{d}{dt}\varphi_k + P_k\right) dq_k. \qquad (2.25)$$

2. Canonical equations. Suppose the generalized usual forces possess the generalized kinetic potential $U(q_i, \dot{q}_i, t)$, i.e.

$$Q_i = \frac{\partial U}{\partial q_i} - \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i} \quad (i = 1, 2, \ldots, S). \qquad (2.26)$$

In this case when conditions (2.1) - (2.3) are satisfied the general equation in the mechanics of a variable mass (8.2) in chpater II for the holonomic systems may be written in the form

$$\delta L' = \frac{d}{dt}\sum_{i=1}^{s} \frac{\partial L'}{\partial \dot{q}_i}\delta q_i, \qquad (2.27)$$

where $L' = T + U + \Theta$ is known as the generalized Lagranges function.

Integrating (2.27) from t_1 to t_2, we get

$$\delta \int_{t_1}^{t_2} L'dt = \sum_{i=1}^{s} p_i' q_i \delta\Big|_{t_1}^{t_2}, \quad p_i' = \frac{\partial L'}{\partial \dot{q}_i}. \qquad (2.28)$$

Take $\delta q_i\Big|_{t_1} = \delta q_i\Big|_{t_2} = 0$, then we have

$$\delta \int_{t_1}^{t_2} L'dt = 0. \qquad (2.29)$$

(2.29) expresses the general principle of Hamilton

9 In the mechanics of a special theory of relativity the Lagranges function is determined artificially (without taking into consideration the force function is potential). It is simply constructed in the form $-m_0c^2\sqrt{1-\beta^2}$ and then we verify the relation

$$\int_{t_1}^{t_2} \left[\delta(-m_0c^2\sqrt{1-\beta^2}) + \bar{F}\delta\bar{r}\right]dt = 0. \qquad (2.30)$$

We find the surplus forces for the point of variable mass if its mass and the reaction force acting on it are determined by the formulae (7.2) and (7,3) in chapter II. As we have seen, in that case $\bar{D} = -\frac{\bar{v}}{2}\frac{dm}{dt}$, therefore, by (8.3) in chapter II we get

$$P_i = \frac{d}{dt}\left(\frac{v^2}{2}\frac{\partial m}{\partial \dot{q}_i}\right) \quad (i = 1, 2, 3). \qquad (2.31)$$

By (7.9) in chapter, we have

$$-\frac{v^2}{2}\frac{\partial m}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(\frac{mv^2}{2} - mc^2\right). \qquad (2.32)$$

Applying (2.31) and (2.32) we obtain

$$P_i = -\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\frac{mv^2}{2} - mc^2\right). \qquad (2.33)$$

We may consider the function $\frac{mv^2}{2} - mc^2$ in place of the function Θ. It is evident that in the mechanics of the theory of relativity, there exists a special Lagranges function, namely

$$L_1 = \frac{mv^2}{2} + \Theta = mv^2 - mc^2,$$

or

$$L_1 = - m_o c^2 \sqrt{1 - \beta^2} \qquad (2.34)$$

Hamilton's principle gives the Lagranges equations with the Lagranges functions

$$\frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = 0 \qquad (2.35)$$

and the canonical equations

$$\frac{d}{dt} q_i = \frac{\partial H'}{\partial p_i}, \quad \frac{d}{dt} p_i' = - \frac{\partial H'}{\partial q_i}, \qquad (2.36)$$

Where the generalized Hamiltons function and the impulses are equal to

$$H' = \sum_{i=1}^{s} p_i' \dot{q}_i - L', \quad p_i' = \frac{\partial L'}{\partial \dot{q}_i}. \qquad (2.37)$$

When the conditions (2.1) - (2.3) and (2.26) are satisfied, an equation in Hamilton Jacobic partial derivatives may be constructed and Jacobic method of the integration may be used. In view of the formula (2.28) for variation in the action function and the canonical equations (2.36) it is possible to generalize other results of the holonomic system and, in particular, the theory of integral invarianes and the contact transformations.

3. Motion of the Holonomic System with Constraints, Depending on the Process of Change of the Mass.

The motion of the mechanical systems with constaints, depending on the process of change of the mass was discussed by the author in

his work [70].

1. Equation of motion: If the holonomic constraints are imposed on a mechanical system, they evidently depend on the cartesian coordinates, time and the mass of points. If masses can be represented by the functions of the cartesion coordinates and time, then we obtain new equations of the holonomic constraints in which masses will be substituted by the functions.

Let us choose a set of independent Lagrangian coordinates q_i (i = 1, 2, ..., S) identically satisfying the new equations of the constraints. In short the constraints are assumed to be ideal. The general equation of the mechanics of a variable mass (8.1) in chapter II, viz.,

$$\sum_j (m_j \bar{a}_j - \bar{F}_j - \bar{R}_j)\ \delta \bar{r}_j = 0,$$

in which the reaction forces $\bar{R}_j$ consist of the impulsive and carioliso forces and the constraints with relative accelerations is written as

$$\sum_{i=1}^{s} \left[\sum_j \frac{m_j}{2} \left(\frac{d}{dt} \frac{\partial v_j^2}{\partial \dot{q}_i} - \frac{\partial v_j^2}{\partial q_i} \right) - Q_i' - \Psi_i' \right] \delta q_i = 0.$$

where $\Psi_i' = \sum_j \bar{R}_j \frac{\partial \bar{r}_j}{\partial q_i}$ are called the generalized reaction forces. The prime on the symbol shows that in order to compute the generalized forces it is necessary to take into account the virtual work caused by the dependence of the constraints of the process of change of the mass. Thim gives us

$$\sum_{i=1}^{s} \left(\frac{d^*}{dt}\frac{\partial^* T_D}{\partial \dot{q}_i} - \frac{\partial^* T_D}{\partial q_i} - Q_i' - \Psi_i'\right) \delta q_i = 0. \qquad (3.1)$$

Here $T_D\ (m_j,\ q_i,\ \dot{q}_i,\ t)$ denotes the kinetic energy of the system.

$$T_D = \frac{1}{2}\sum_j m_j v_j^2\ (\dot{q}_i,\ q_i,\ t). \qquad (3.2)$$

As has already been said $\frac{\partial^*}{\partial \dot{q}_i}$ and $\frac{\partial^*}{\partial q_i}$ are the partial derivatives for these indicated variables and $\frac{d^*}{dt}$ is the total derivative in respect of time. These are calculated under the steady mass m_j as expressed in (3.2). Considering that δq_i is independent of (3.1), we can obtain the lagranges equation of the second type in the form

$$\frac{d^*}{dt}\frac{\partial^* T_D}{\partial \dot{q}_i} - \frac{\partial^* T_D}{\partial q_i} = Q_i' + \Psi_i' \quad (i = 1,\ 2,\ \ldots,\ s). \qquad (3.3)$$

In terms of regular derivatives the above equations can be written as

$$\frac{d}{dt}\frac{\partial T_D}{\partial \dot{q}_i} - \frac{\partial T_D}{\partial q_i} = Q_i' + P_i',$$

where the additional generalized forces are equal to

$$P_i' = \sum_j \left[\left(R_j + \frac{dm_j}{dt}\bar{v}_j\right)\frac{\partial \bar{r}_j}{\partial q_i} - \frac{v_i^2}{2}\frac{\partial m_j}{\partial q_i}\right].$$

2. Example: As an illustrating example let us investigate the rolling inertia along the horizontal rough plane of a three wheeled trolly whose wheels are made of sparkling and absolutely flexible strip (land) of thickness h and the linear density γ. Suppose the planes of wheels stay parallel all along the motion. In such a case the trolly will move in a straight line along a certain

axis x.

The kinetic energy of the trolly is given by

$$T = \frac{m(\dot{x}^2 + \dot{z}^2)}{2} + \frac{3I}{2}\frac{\dot{x}^2}{z^2},$$

where m is mass of the trolly, I is the moment of inertia of our wheel for the axis of revolution; z is radius of the wheel.

If the unwind when the trolly rolls, the mass of the wheel is given by

$$\frac{dm}{dt} = -\gamma \dot{x}, \quad m = \gamma(l - x)$$

where l is the initial length of . It is easy to see that

$$m = \pi \frac{\gamma}{h} z^2.$$

we get

$$z^2 = \frac{h}{\pi}(l - x), \quad \dot{z} = -\frac{1}{2}\sqrt{\frac{h}{\pi}}\frac{\dot{x}}{\sqrt{l - x}}.$$

The system under consideration has one degree freedom. Consider the coordinate x in place of the lagrangian coordinate. Then formula (3.2) gives

$$T_D = \frac{m(1 + \frac{1}{4}\frac{h}{l - x}\frac{1}{l - x})\dot{x}^2}{2} + \frac{3I}{2}\frac{\pi}{h}\frac{\dot{x}^2}{l - x}.$$

The particles of the land which get wound, leave with a relative velocity of zero. Moreover, there is no internal motion of the particles.

As such the reaction forces are not generated. The forces of weight characterize the active forces. If the radius of the wheels do not change then the virtual work done by the force (of weight) is equal to zero. In this case

$$- mg\,\delta z = -\frac{mg}{2}\sqrt{\frac{h}{\pi}}\,\frac{1}{\sqrt{1-x}}\,\delta x,$$

therefore the generalized forces give

$$Q' = \frac{mg}{2}\sqrt{\frac{h}{\pi}}\,\frac{}{\sqrt{1-x}}\,.$$

For the coordinate X, let us construct equations analogous to (3.3)

$$m\,\frac{d}{dt}\left[\left(1+\frac{1}{4}\,\frac{a}{\pi}\,\frac{1}{1-x}\right)\dot{x}\right] + 3I\,\frac{\pi}{h}\,\frac{d}{dt}\,\frac{\dot{x}}{1-x} -$$

$$-\frac{1}{2}\left(\frac{mh}{4\pi}+3I\,\frac{\pi}{h}\right)\frac{\dot{x}^2}{(1-x)^2} = \frac{mg}{2}\sqrt{\frac{h}{\pi}}\,\frac{1}{\sqrt{1-x}}\,.$$

In short, let us neglect the mass of the trolley

$$m = m_{trolley} + 3m_b = m_{trolley} + 3\gamma(1-x),$$

$$I = \frac{m_b z^2}{2} = \frac{\gamma h}{2\pi}(1-x)^2.$$

then we obtain

$$\frac{d\dot{x}^2}{dx} + \frac{\left[h+2\pi(1-x)\right]\dot{x}^2}{(1-x)\left[h+6\pi(1-x)\right]} = 4g\sqrt{\pi h}\,\frac{\sqrt{1-x}}{h+6\pi(1-x)}. \qquad (3.4)$$

integrating (3.4) we get

$$\dot{x}^2 = \frac{1-x}{\left[h+6\pi(1-x)\right]^{\frac{2}{3}}}\left[\frac{\dot{x}_o^2(h+6\pi 1)^{\frac{2}{3}}}{1} - 8g\sqrt{\pi h}\int_{\sqrt{1}}^{\sqrt{1-x}}\frac{dy}{3\sqrt{h+6\pi y^2}}\right].$$

3. The principle of consolidation for the holonomic systems with constraints, depending on the change of mass: **The earlier discussion shows that the lagrangian equations of the second type (3.3) can be adopted to the corresponding system with constant masses if the system preserves the complex kinematic of the motion due to the change of mass, and if the reaction forces also act on the system**

Hence, for holonomic systems, dependent on the process of changes in mass, the mechanical system under consideration, will serve as a state of solidification, where beginning from the given time t, the process of change in mass ceases and where the same kinematics of motion are retained (the same changable connections) as the system with variable mass under study.

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4. The System of a Fixed Composition with Realtive Motion of the Particles Considered as a Body with a Variable Mass.

The next three examples will illustrate that it is possible to consider a system with a fixed composition and the relative motion of the particles as a body with a variable mass. The Lagrangian equations in respect of a body with a variable mass will describe the motion of the frame of the system.

1. First example: Let a tube of the mass m_1 and length l_1 revolve on a vertical plane about its own end O. There passes a rod of mass m_2 and length l_2 through the table. This arrangement makes a holonomic system with two degree freedom. Take the angle of the revolution of the tube and the relative displacement (translocation) S of the rod as the Lagrangian coordinates. Then the kinetic energy is given by

$$T = \frac{1}{2}\left\{ m_2 \dot{S}^2 + \left[\frac{m_1 l_1^2}{3} + m_2 \left(S + \frac{l_2}{2}\right)^2 + \frac{m_2 l_2^2}{12} \right] \dot{\varphi}^2 \right\}. \tag{4.1}$$

The potential energy is

$$\Pi = -\left[\frac{1}{2} m_1 l_1 + m_2 \left(S + \frac{l_2}{2}\right) \right] g \sin\varphi . \tag{4.2}$$

We then obtain the usual Lagrangian equations of the second type

$$\left[\frac{m_1 l_1^2}{3} + m_2 \left(S + \frac{l_2}{2}\right)^2 + \frac{m_2 l_2^2}{12} \right] \ddot{\varphi} + 2m_2 \left(S + \frac{l_2}{3}\right) \dot{S} \dot{\varphi} =$$

$$= \left[\frac{m_1 l_1}{2} + m_2 \left(S + \frac{l_2}{2}\right) \right] g \cos\varphi , \tag{4.3}$$

$$\ddot{S} - (S + \frac{l_2}{2})\,\varphi^2 = g \sin\varphi . \qquad (4.4)$$

This problem can be looked upon as the problem of rotation about a fixed axis of the body. Although the mass of the body on the whole remains unchanged, but certain geometrical points gain additional mass, while others lose it.

Such a body with a variable mass has only one degree of freedom. The angle φ will work as the Lagrangian coordinate. The kinetic energy of the body is

$$T = \frac{1}{2}\left[\frac{m_1 l_1^2}{3} + m_2 (S + \frac{l_2}{2})^2 + \frac{m_2 l_2^2}{12}\right]\dot{\varphi}^2 . \qquad (4.5)$$

The potential energy, as before, is given in the formula (4.2). Set up Lagrangian equation of the second type

$$\frac{d^*}{dt}\frac{\partial^* T}{\partial \dot{\varphi}} - \frac{\partial^* T}{\partial \varphi} = Q_\varphi + \overline{Q}_\varphi , \qquad (4.6)$$

where $\overline{Q}_\varphi$ is the reaction force. The star means that for the computation of the derivatives it may be supposed that the body with a variable mean considered as partial derivatives in which the variable S is regarded constant. Therefore, we write

$$\frac{d^*}{dt}\frac{\partial^* T}{\partial \dot{\varphi}} - \frac{\partial^* T}{\partial \varphi} = \left[\frac{m_1 l_1^2}{3} + m_2 (S + \frac{l_2}{2})^2 + \frac{m_2 l_2^2}{12}\right]\ddot{\varphi} . \qquad (4.7)$$

The generalized force is equal to

$$Q_\varphi = \left[\frac{m_1 l_1}{2} + m (S + \frac{l_2}{2})\right] g \cos\varphi . \qquad (4.8)$$

The generalized reaction force characterizes the moment of reaction forces for the point O. We will thus have

$$M = \sum \bar{r} \times \Phi - \sum \bar{r} \times m \bar{w}_r - \sum \bar{r} \times m \bar{w}_c. \qquad (4.9)$$

The impulse force Φ is equal to zero on account of the continuous change in the velocity (speed) of the rod. The relative acceleration of the rod is directed along the axis of the tube and thus the moment of this acceleration for the point O becomes zero. The moment of the reaction force is made up of the carioliso moment only.

$$M = -\sum m (\bar{r} \times \bar{w}_c), \; \bar{w}_c = 2 \bar{\omega} \times \bar{v}_r.$$

Since $v_r = \dot{S}$, $\omega = \dot{\varphi}$, we get

$$\Psi = -2\sigma S \dot{\varphi} \int_0^{l_2} (S + y)\, dy, \quad \sigma = m_2 l_2^{-1}.$$

Integrating we obtain

$$\Psi_\varphi = -2m_2 S \dot{\varphi} \left(S + \frac{l_2}{2}\right). \qquad (4.10)$$

Applying (4.7), (4.8) and (4.10), equation (4.6) takes the form (4.3). In the mechanics of the systems with variable masses, internal motion of the particles is a common feature. The conditions of the problem are such that they will make the rod move under the action of gravity and the centrifugal force caused by the rotation of the body. The realtive displacement of the rod is given by the equation.

$$\ddot{S} = g \sin \varphi + \left(S + \frac{1}{2}\right) \dot{\varphi}^2. \qquad (4.11)$$

Which is the same as equation (4.4).

2. Second Example: A hallow circular cylinder half filled with ideal liquid revolves round a horizontal axis which is the same as the axis of the cylinder. Let I_1, be the inertial moment of the cylinder and I_2 be the inertial moment of the liquid. When the cylinder revolves the ideal liquid remains motionless, therefore $T = \frac{1}{2} I_1 \dot{\varphi}_2$ and the usual Lagranges equation of the second type is wirtten in the form

$$I_1 \ddot{\varphi} = Q_\varphi . \tag{4.12}$$

The points of a body with a variable mass are regarded as small regions (geometrical points) of a fixed system firmly connected with the frame of the body. The kinetic energy of the body is given by

$$T = \frac{1}{2} (I_1 + I_2) \dot{\varphi}^2 .$$

The Lagranges equation of the second type is

$$(I_1 + I_2) \ddot{\varphi} = Q_\varphi + \Psi_\varphi . \tag{4.13}$$

The liquid remains motionless in respect of an absolute space, therefore, the relative velocity is equal to a portable velocity taken with a negative sign.

Because the Coriolis and centrifugal accelerations pass through the axis of rotation, the moment of reactive forces are the total particles from the axis of rotation and tangent compounds of reactive accelerations, taken with a negative sign and multiplied by the particles of vectorial product of distances (vectors).

We thus get the basis vector

$$\bar{\Psi} = I_2 \dot{\varphi}. \qquad (4.14)$$

Formulae (4.13) and (4.14) yield the equation of motion (4.12).

3. Third example: We will investigate the rotation of a glass partially filled with liquid about the horizontal axis which is perpendicular to the axis of the glass. When the glass revolves relative velocities from a complex field. The liquid particles which are close to the center of the boundary surface of the liquid shifts in regard to a fixed space, therefore, the particles which are close to the bottom of the glass have zero relative velocities. The velocity of the liquid particle is equal to the sum of the portable velocity $\bar{v}_e$, the present velocity of revolution of the corresponding point in the rigid system and the relative velocity $\bar{v}_r$. The square of the velocity is equal to

$$v^2 = v_e^2 + v_r^2 + 2v_e v_r \cos(\bar{v}_e, \hat{\ } \bar{v}_r).$$

In short, suppose that the cross section of the glass in a rectangle and the relative velocity $\bar{v}_r$ perpendicular to the axis of revolution. Let $\bar{r}$ denote the position vector of the point where the axis of revolution meets the plane section of the glass, which is perpendicular to the axis. Since $\cos(\bar{v}_e, \hat{\ } \bar{v}_r)$ is equal to $\cos(90^\circ + \bar{r}, \hat{\ } \bar{v}_r)$, and $v_e = r\dot{\varphi}$, then

$$v^2 = v_e^2 + v_r^2 + 2\dot{\varphi}(\bar{r} \quad \bar{v}_r)\,\bar{b}.$$

Here $\bar{b}$ is the axis of revolution of the glass.

The kinetic energy of the glass and the liquid is wirtten as

$$T = \frac{1}{2} I(\varphi)\, \dot{\varphi}^2 + \frac{1}{2} \iiint \rho v_r^2 d\tau + \dot{q} \iiint \rho(\bar{r}\ \bar{v}_r)\, \delta\, d\tau.$$

$I(\varphi)$ represents the moment of the inertia of the glass and the liquid in respect of the axis of revolution, ρ is liquid density. The integration is carried over the entire volume of the liquid.

The glass has one degree freedom and its position determines the angle of rotation q. The liquid possesses infinite degree of freedom. The relative Lagrangian velocities (possible coordinates) enter into the kinetic energy through.

The usual Lagrangian equation of the second type for the coordinate is written as q

$$\frac{d}{dt}\left[I(q)\dot{q}\right] + \frac{d}{dt}\iiint \rho(\bar{r}\ \bar{v}_r)\, \delta\, d\tau -$$

$$- \frac{1}{2}\frac{dI}{dq}\dot{q}^2 - \dot{q}\iiint \rho\left(\frac{d\bar{r}}{dq}\ \bar{v}_r\right) \delta\, d\tau = Q_q.$$

Last term of the laft hand side can also be written in the form of

$$- \iiint \rho\left(\frac{d\bar{r}}{dt}\ \bar{v}_r\right) \delta\, d\tau.$$

Thus the equation of the motion is obtained in the form of

$$I\dot{q} + \frac{1}{2}\frac{dI}{dq}\dot{q}^2 + \iiint \rho\left(\bar{r}\ \frac{d\bar{v}_r}{dt}\right) \delta\, d\tau = Q_q. \quad (4.15)$$

Now we can consider the glass with liquid as a body with a variable mass. The Lagrangies equation of the mechanics of a variable mass will take the form

$$I\ddot{\varphi} = Q_\varphi + \Psi_q. \quad (4.16)$$

The generalized reaction force $\bar{Q}_\varphi$ includes moments of Carioliso, moments of impulsive force and of those called relative accelerations.

The Carioliso moment is equal to

$$\bar{R} = -\sum m\bar{r} \quad (2\bar{\omega} \times \bar{v}_r) = -2\sum m\bar{\omega}\,(\bar{r}\,\bar{v}_r) + 2\sum m\bar{v}_r\,(\bar{\omega}\,\bar{r}).$$

Here $\bar{\omega} = \dot{\varphi}\,\bar{b}$. Since the last sum on the right hand side is equal to zero, we get

$$\bar{R} = -2\sum m\bar{\omega}\,(\bar{r}\,\bar{v}_r).$$

We know that $I = \sum m\,\bar{r}\,\bar{r}$. Therefore $\frac{dI}{d\varphi} = 2\sum m\bar{r}\bar{v}_r \frac{dt}{d\varphi}$. Here

$$\bar{R} = -\frac{dI}{d\varphi}\dot{\varphi}^2\bar{b}. \qquad (4.17)$$

Now consider the moment of reaction force and of those called relative accelerations. For relative acceleration we write the known formula

$$\bar{\omega}_r = \frac{d\bar{v}_r}{dt} - \bar{\omega} \times \bar{v}_r. \qquad (4.18)$$

For the remaining part of the moment of reaction forces, use the expression

$$-\sum \bar{r} \times m\bar{\omega}_r = -\sum \bar{r} \quad m\frac{d\bar{v}_r}{dt} + \sum m\bar{r} \times (\bar{\omega} \times \bar{v}_r). \qquad (4.19)$$

As in the case of moment of carioliso forces, the second term on the right hand side is directed in the positive direction of the axis of revolution and is equal to

$$\sum m\bar{r} \times (\bar{\omega} \times \bar{v}_r) = \frac{1}{2}\frac{dI}{d\varphi}\dot{\varphi}^2\bar{b}. \qquad (4.20)$$

Using formulae (4.17), (4.19) and (4.20), we find that

$$\dot{U}_{\varphi} = -\frac{1}{2}\frac{dI}{d\varphi}\dot{\varphi}^2 - \sum (\bar{r} \times m \frac{d\bar{v}_r}{dt})\,\bar{b}. \qquad (4.21)$$

(4.21) can be used to transform (4.16) to the form (4.15). Writing equation of the motion by the method of the mechanics of a variable mass is simple.

The liquid flowing out of the glass does not affect the additional reaction forces; for in this case there is no impulsive change of velocity in the particles. Therefore, the equation of motion (4.15) also holds in this case.

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CHAPTER-IV

NON-HOLONOMIC SYSTEMS WITH VARIABLE MASSES

1. Fundamental Equations of Motion

1. Equations with undetermined multipliers of reactions of the ideal non holonomic constraints in the mechanics of a variable mass. Suppose the motion of a mechanical system whose position is completely determined by S generalized or the Lagrangian coordinates: $q_1, q_2, \ldots, q_s$ is subjected to r generalised velocity constraints which are linear and non-holonomic.

$$F_k = \sum_{i=1}^{s} a_{ki}\dot{q}_i + a_{ko} = 0, \quad a_{kj} = a_{kj}(q_i, t)$$

$$(k = 1,2, \ldots, r; \; j = 0, 1, \ldots, s). \qquad (1.1).....$$

The equations of motion of the systems (with variable masses) subject to non-linear and non-holonomic constraints were discussed by the author in this works (64).

The constraints (1.1) imposte the following conditions on the variations of the generalised coordinates.

$$\sum_{i=1}^{s} a_{ki}\delta q_i = 0 \qquad (1.2).....$$

or

$$\sum_{i=1}^{s} \frac{\partial F_k}{\partial q_i} \delta q_i = 0. \qquad (1.3).....$$

Therefore, only $l = s-r$, variations will be independent.

Multiply relations (1.3) by undetermined multipliers λ_k, take their sum from 1 to r and subtract from the general equation (8.2) of chapter II. We thus get

$$\sum_{i=1}^{s} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i - P_i - \sum_{k=1}^{r} \lambda_k \frac{\partial F_k}{\partial q_i} \right) \delta q_i = 0.$$

$$(1.4).....$$

Since the equations of constraints (1.1) are independent, the rank of the matrix

$$\left\| \frac{\partial F_k}{\partial q_i} \right\| \quad (k = 1, 2, \ldots, r; \; i = 1, 2, \ldots, s) \qquad (1.5).....$$

is equal to r. The r^{th} order determinant of the matrix (1.5) which is different from zero is in terms of the generalized velocities where $i = l+1, l+2, \ldots, s$. Then the relations (1.3) may be solved in respect of variations $\delta q_{l+1}, \delta q_{l+2}, \ldots, \delta q_s$, which are obtained as linear functions $\delta q_1, \delta q_2, \ldots, \delta q_s$.

Choose $\lambda_1, \lambda_2, \ldots, \lambda_k$ such a way that the expressions in

$\delta q_{l+1}, \delta q_{l+2}, \ldots, \delta q_s$ of equation (1.4), corresponding to the dependent variations λ_k become zero: this situation is the case because the determinant with - is different from zero. In view of the independence of the variations $\delta q_1, \delta q_2, \ldots, \delta q_l$ the expressions in the remaining terms of equation (1.4) must be zero. Thus we obtain equations with undetermined multipliers in the form of

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + P_i + \sum_{k=1}^{r} \lambda_k \frac{\partial f_k}{\partial q_i} \quad (i = 1, 2, \ldots, s). \qquad (1.6) \ldots\ldots$$

The equations (1.6) will be solved together with the constraints equations (1.1).

The terms

$$\Lambda_i = \sum_{k=1}^{r} \lambda_k \frac{\partial f_k}{\partial q_i} \qquad (1.7) \ldots\ldots$$

are called the generalized forces of the reaction constraints.

The values of λ_k ($k = 1, 2, \ldots, r$), can be obtained as an explicit expression. Express the constraint equation in terms of the canonical variables q_i and p_i, where

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \quad (i = 1, w, \ldots, s).$$

We obtain r constraint equations in the form

$$\tilde{F}_k (q_i, p_i, t) = 0. \qquad (1.7)$$

Differentiating (1.7) we get

$$\sum_{i=1}^{s} \frac{\partial \tilde{F}_k}{\partial p_i} \dot{p}_i + \sum_{i=1}^{s} \frac{\partial \tilde{F}_k}{\partial q_i} \dot{q}_i + \frac{\partial \tilde{F}_k}{\partial t} = 0.$$

In this put the values of $\dot{p}_i$ from (1.6).

$$\sum_{i=1}^{s} \frac{\partial \tilde{F}_k}{\partial p_i} \left(\frac{\partial T}{\partial q_i} + Q_i + P_i + \sum_{\mu=1}^{r} \lambda_\mu \frac{\partial f_\mu}{\partial q_i} \right)$$

$$+ \sum_{i=1}^{s} \frac{\partial \tilde{F}_k}{\partial q_i} \dot{q}_i + \frac{\partial \tilde{F}_k}{\partial t} = 0.$$

This gives

$$\sum_{\mu=1}^{r} \lambda_\mu A_{k\mu} = - \frac{D\hat{F}_k}{Dt} . \qquad (1.9).....$$

$$A_{k\mu} = \sum_{i=1}^{s} \frac{\partial \hat{F}_k}{\partial p_i} \frac{\partial F_\mu}{\partial q_i} , \qquad (1.10).....$$

$$\frac{D\hat{F}_k}{Dt} = \sum_{i=1}^{s} \frac{\partial \hat{F}_k}{\partial p_i} \left(\frac{\partial T}{\partial q_i} + Q_i + P_i \right) + \sum_{i=1}^{s} \frac{\partial \hat{F}_k}{\partial q_i} \dot{q}_i + \frac{\partial \hat{F}_k}{t} .$$

$$(1.11).....$$

The values $\frac{\hat{F}_k}{Dt}$ are identified as derivatives of the constraint equations in case of free motion, that is, motion without the influence of the constraints (1.1). On the other hand, if the constraints act on the motion, then $\frac{D\hat{F}_k}{Dt}$ would represent the ordinary derivatives in respect of time and will tend to zero.

Let us suppose that the determinant of the matrix $A_{k\mu}$ is different from zero. We will show that the mechanical systems, subject to the constraints of the type (1.1) this determinant is always different from zero.

The expression (1.10) can be re-written in the form

$$A_{k\mu} = \sum_{i,j=1}^{s} \frac{\partial \hat{F}_\mu}{\partial p_j} \frac{\partial p_j}{\partial q_i} \frac{\partial \hat{F}_k}{\partial p_i} . \qquad (1.12).....$$

Therefore, the matrix of coefficients $A_{k\mu}$ can be written as the product of the three matrices 1 viz. matrix $\left\| \frac{\partial F_\mu}{\partial p_j} \right\|$ having r rows and s columns, square matrix

$$\left\| \frac{\partial p_i}{\partial q_i} \right\| = \left\| \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i} \right\| \quad (i,j = 1, 2, \ldots, s),$$

which is known as a positive quadratic form, and finally a matrix with s rows and r columns which is the transform of the matrix $\left\| \frac{\partial \hat{F}_\mu}{\partial p_j} \right\|$.

The rank of the matrix $\left\| \frac{\partial \hat{F}_\mu}{\partial p_j} \right\|$ is equal to r which follows from the linear dependence of

$$\frac{D(F_1, F_2, \ldots, F_r)}{D(q_{i_1}, q_{i_2}, \ldots, q_{i_r})} = \sum_{1 \le j_1 < j_2 < \cdots < j_r \le s} \cdots \times$$

$$\times \frac{D(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_r)}{D(p_{j_1}, p_{j_2}, \ldots, p_{j_r})} \quad \frac{D(p_{j_1}, p_{j_2}, \ldots, p_{j_r})}{D(\dot{q}_{i_1}, \dot{q}_{i_2}, \ldots, \dot{q}_{i_r})} \qquad (1.13).....$$

Since, by supposition, F_k are linearly independent in respect of generalized velocity functions, then for $q_{i_1}, q_{i_2}, \ldots, q_{i_r}$ the determinant

$$\frac{D(F_1, F_2, \ldots, F_r)}{D(q_{i_1}, \dot{q}_{i_2}, \ldots, \dot{q}_{i_r})}$$

will be different from zero. Therefore, on the basis of (1.13) it follows that the determinant $p_{j_1}, p_{j_2}, \ldots, p_{j_r}$ for

$$\frac{D(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_r)}{D(p_{j_1}, p_{j_2}, \ldots, p_{j_r})}$$

must be different from zero.

Further, making the transformation

$$q_i = \sum_{k=1}^{r} \frac{\partial \tilde{F}_k}{\partial p_i} \quad k(i = 1,2, \ldots, s),$$

we find that the positive quadratic form

$$\sum_{i,j=1}^{s} \frac{\partial^2 T}{\partial \dot{q}_j \partial \dot{q}_i} \dot{q}_j \dot{q}_i$$

is equivalent to the quadratic form

$$\sum_{k,\mu=1}^{r} A_{k\mu} U_k U_\mu \quad . \qquad (1.14).....$$

Since the rank of the matrix

$$\left\| \frac{\partial \hat{F}_k}{\partial p_i} \right\|^r \quad (k=1, 2, \ldots, r;\ i=1, 2, \ldots, s)$$

is equal to r, the form (1.14) will be positively determined and the determinant of matrix $\| A_{k\mu} \|^r$ is different.

Solving equations (1.9) for $\dot{p}_\mu$, we get

$$\dot{p}_\mu = - \sum_{k=1}^{r} B_{\mu k} \frac{\partial \hat{F}_k}{\partial t} =$$

$$= - \sum_{k=1}^{r} B_{\mu k} \left(\sum_{i=1}^{s} \frac{\partial \hat{F}_k}{\partial p_i} \left(\frac{\partial T}{\partial q_i} + Q_i + P_i \right) + \right.$$

$$\left. + \sum_{i=1}^{s} \frac{\partial \hat{F}_k}{\partial q_i} \dot{q}_i + \frac{\partial \hat{F}_k}{\partial t} \right) . \qquad (1.15).....$$

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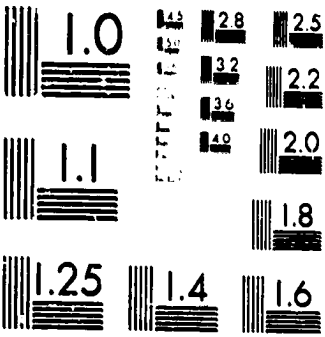

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In this, $\lambda_{\mu k}$ denote the elements of the inverse matrix (inverse of $\|a_{k\mu}\|$.

Formula (1.15) expresses λ_μ ($\mu = 1, 2, \ldots, r$) in terms of q_i, $\dot{q}_i$, t and p_i. The explicit expressions are obtained with the help of (1.7) for the generalized forces of the reactions of the non-holonomic constraints R_{-i}. The values of R_{-i} like those of p_i will depend upon time, the Lagrangian coordinates and the velocities and linearly on the second order derivatives of the Lagrangian coordinates. If the sum of forces $Q_i + P + R_{-i}$ satisfies the conditions of existence of the kinetic potential, then equations (1.6) can be written in the Lagrangian form. Equations (1.6) describe the motion of a holonomic system with variable masses which change, as in a non-holonomic system, if the kinetic energy T of the system and the sum of the generalized and surplus forces is equal to $Q_i + P_i + R_i$. Only those types of motions whose initial conditions satisfy the relations

$$F_k(q_i^o, \dot{q}_i^o, t_o) = 0.$$

are characterized by trajectories of non-holonomic system.

When the conditions of the existence of the generalized kinetic potential are satisfied, we may apply the theory of the canonical equations, the method of Jacobic integration and make the expression of the integral invariants for auxiliary holonomic system. However, for obtaining results in the case of non-

holonomic system it is necessary to take into account (60).

2. Equation of Motion in Mechanics of Variable Mass.

Suppose the constraint equations (1.1) identically satisfy the linear relations.

$$q_i = \sum_{\nu=1}^{l} \beta_{i\nu} \pi_\nu + \beta_{i0}, \quad \beta_{ij} = \beta_{ij}(q_1, q_2, \ldots, q_s, t) \qquad (1.16).....$$

$(i = 1, 2, \ldots, s;\ j = 0, 1, 2, \ldots, l)$,

where π_ν - are certain kinetic responses (characteristics).

$$\delta q_i = \sum_{\nu=1}^{l} \beta_{i\nu} \delta \pi_\nu \qquad (1.17).....$$

or

$$\delta q_i = \sum_{\nu=1}^{l} \frac{\partial q_i}{\partial \pi_\nu} \delta \pi_\nu . \qquad (1.18).....$$

Variations $\delta \pi_1, \delta \pi_2, \ldots, \delta \pi_l$ are independent.

Proceeding from the general equation of the mechanics (8.2) in Chapter II, we get l symmetric equations in the form

$$\sum_{i=1}^{s} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} - Q_i - I_i \right) \frac{\partial q_i}{\partial e_\nu} = 0 \quad (\nu = 1, 2, \ldots, l). \qquad (1.19).....$$

Uniting (1.19) and (1.16) we obtain a system containing $s + l$ equations in $s + l$ variables: $e_1 e_2, \ldots, e_l; q_1, q_2, \ldots, q_s$.

3.

Equations in the mechanics of variable mass. We will denote by T the kinetic energy given in terms of e_ν, q_i, $\dot{e}_\nu$, $\dot{q}_i$ ($\nu = 1, 2, \ldots, l$; $i = 1, 2, \ldots, s$) by the relations (1.16). We then obtain

$$\frac{\partial T}{\partial e_\nu} = \sum_{j=1}^{s} \frac{\partial T}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial e_\nu}, \qquad (1.20).....$$

$$\frac{\partial T}{\partial q_i} = \frac{\partial T}{\partial q_i} + \sum_{j=1}^{s} \frac{\partial T}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i}. \qquad (1.21).....$$

Equations (1.19) are written in the form

$$\frac{d}{dt} \sum_{i=1}^{s} \frac{\partial T}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial e_\nu} - \sum_{i=1}^{s} \frac{\partial T}{\partial \dot{q}_i} \left(\frac{d}{dt} \frac{\partial \dot{q}_i}{\partial e_\nu} - \frac{\partial \dot{q}_i}{\partial e_\nu} \right) = E_\nu + G_\nu,$$

(1.22).....

where

$$E_\nu = \sum_{i=1}^{s} \ldots \quad , \quad G_\nu = \sum_{i=1}^{s} P_i f'_{i\nu} \qquad (1.\ldots)\ldots\ldots$$

are generalized active and additional forces responsible for the kinetic responses $\ldots$.

Taking into consideration (1.20) and (1.21), equations (1.23) can be written as

$$\frac{d}{dt}\frac{\partial \hat{T}}{\partial \dot{e}_\nu} - \sum_{i=1}^{s} \frac{\partial \hat{T}}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{e}_\nu} -$$

$$- \sum_{i=1}^{s} \frac{\partial T}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial \dot{q}_i}{\partial q_\nu} - \sum_{j=1}^{s} \frac{\partial \dot{q}_i}{\partial q_j}\frac{\partial q_j}{\partial e_\nu}\right) = E_\nu + G_\nu \ .$$

(1.24).....

Let us put

$$\frac{\partial}{\partial \ldots_\nu} = \sum_{i=1}^{s} \frac{\partial \dot{q}_i}{\partial \dot{e}_\nu}\frac{\partial}{\partial \dot{q}_i} \ , \qquad (1.25).....$$

where the partial derivatives in respect of q_i are computed for the set of variables

$$e_\nu \ , \ q_i \ , \ t. \qquad —$$

Therefore, we get

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{e}_\nu} - \frac{\partial T}{\partial \pi_\nu} - \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial \dot{q}_i}{\partial \dot{e}_\nu} - \frac{\partial \dot{q}_i}{\partial \pi_\nu}\right) = E_\nu + G_\nu \quad .$$

(1.26).....

omitting the lagrangian operator.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{e}_\nu} - \frac{\partial}{\partial \pi_\nu}$$

Using L_{e_ν} , for the lagrangian operator, we write (1.26) in a more compact form.

$$L_{e_\nu}(T) - \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} L_e(\dot{q}_i) = E_\nu + G_\nu \quad .$$

The coefficients

$$L_{e_\nu}(\dot{q}_i) = \frac{d}{dt}\frac{\partial \dot{q}_i}{\partial \dot{e}_\nu} - \frac{\partial \dot{q}_i}{\partial \pi_\nu} \qquad (1.27).....$$

are known as kinematic values. These values depend only on the selection of the kinetic responses (characteristics) and not on the character of the mechanical system. These will be called **Chapligin's coefficients.**

Equation(1.26) we shall call Chaplygin's equation in the mechanics of variable mass. For the integration of (1.26) combine with it relations (1.16) substituting the constraints. As a result, we get a system of $s + 1$ differential equations of the first order in $s + 1$ variable $\dot{q}_\nu$, q_i.

Since the constraint equations are solved in terms of derivatives of dependent lagrangian coordinates $\dot{q}_{l+1}, \dot{q}_{l+2}, \ldots, \dot{q}_s$, we will have relations of the type

$$\dot{q}_{l+k} = \sum_{\nu=1}^{l} B_\nu^{(k)} \dot{q}_\nu + B^{(k)} \qquad (1.28).....$$

where the coefficient $B_\nu^{(k)}$, $B^{(k)}$ ($\nu = 1, 2, \ldots, l$; $k = 1, 2, \ldots, r$) functions in variables q_i, t ($i = 1, 2, \ldots, s$).

If for the kinetic energy we take independent lagrangian velocities $\dot{q}_\nu$, then it is necessary to use relation (1.28) in place of (1.16). Making use of the notation provided in (1.25), equation (1.26) can be written in the form

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\nu} - \frac{\partial \tilde{T}}{\partial q_\nu} - \sum_{k=1}^{r} \frac{\partial T}{\partial \dot{q}_{l+k}} \left(\frac{d}{dt}\frac{\partial \dot{q}_{l+k}}{\partial \dot{q}_\nu} - \frac{\partial \dot{q}_{l+k}}{\partial q_\nu} - \right.$$

$$\left. - \sum_{\mu=1}^{r} \frac{\partial \dot{q}_{l+k}}{\partial q_{l+\mu}} \frac{\partial \dot{q}_{l+\mu}}{\partial \dot{q}_\nu} \right) - \sum_{k=1}^{r} \frac{\partial \tilde{T}}{\partial q_{l+k}} \frac{\partial \dot{q}_{l+k}}{\partial \dot{q}_\nu} = {}'Q_\nu + {}'P_\nu \ .$$

(1.29).....

In view of the values (1.23) we have

$$'Q_\nu = Q_\nu + \sum_{k=1}^{r} {}'q_{1+k}\beta_\nu^{(k)} \quad , \quad 'P_\nu = P_\nu + \sum_{k=1}^{r} {}'P_{1+k}\beta_\nu^{(k)} \quad .$$

Suppose the kinetic energy T, the generalized force $'Q_\nu$ and $'P_\nu$ and the relations (1.29) do not (explicitly) contain $q_{1+k}(k = 1, w, \ldots, r)$. In such a case, the mechanical system is called $\tilde{T}$. For the Chaplygin system characteristic function of q_1, q_2, t and the following relations are satisfied.

$$\frac{\partial \tilde{T}}{\partial q_{1+k}} = 0, \quad \frac{\partial \beta_{1+k}}{\partial q_{1+\omega}} = 0 \ (k, \omega = 1, 2, \ldots, r).$$

Consequently, the chaplygin equations (1.26) in the present case take the form of

$$L_{q_\nu}(\tilde{T}) - \sum_{k=1}^{r} \frac{\partial T}{\partial q_{1+k}} L_{q_\nu}(\dot{q}_{1+k}) = 'Q_\nu + 'P_\nu \quad , \qquad (1.30).....$$

where the lagrange operator L_{q_ν} is equal to

$$L_{q_\nu} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_\nu} - \frac{\partial}{\partial q_\nu} \quad .$$

Equations (1.30) describe a closed system of differential equations of the second order in variables $q_1, q_2, \ldots, q_1$.

When the relations (1.28) are stationary and homogeneous, we have

$$L_{q_\nu}(\dot{q}_{l+k}) = \frac{d}{dt} B_\nu^{(k)} - \sum_{\gamma=1}^{l} \frac{\partial B_\gamma^{(k)}}{\partial q_\nu} \dot{q}_\gamma ,$$

have

$$L_{q_\nu}(\dot{q}_{l+k}) = \sum_{\gamma=1}^{l} \left(\frac{\partial B_\gamma^{(k)}}{\partial q_\nu} - \frac{\partial B_\nu^{(k)}}{\partial q_\gamma} \right) \dot{q}_\gamma . \qquad (1.31).....$$

Equations (1.30) become

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\nu} - \frac{\partial T}{\partial q_\nu} + \sum_{k=1}^{r} \sum_{\gamma=1}^{l} \frac{\partial T}{\partial \dot{q}_{l+k}} \dot{q}_\gamma \left(\frac{\partial B_\gamma^{(k)}}{\partial q_\nu} - \frac{\partial B_\nu^{(k)}}{\partial q_\gamma} \right) = Q_\nu + Q_\nu' .$$

(1.32).....

Equations of the type of (1.32) for the mechanics of the constant masses where $F_\nu' = 0$, were obtained by S.A. Chaplygin in the year 1897 (99). Hence the origin of the nomenclature of the above discussed equations.

4. Vorontz-Hamel Equations in the Mechanics for Variable Mass:

Let us introduce kinematic responses (characteristics) by independent relations in terms of the derivatives of the lagranges coordinates.

$$e_\nu = \sum_{i=1}^{s} \alpha_{r+\nu, i} \dot{q}_i + \alpha_{r+\nu, 0}, \quad \alpha_{r+\nu, j} = \alpha_{r+\nu, j}(q_i, t) \qquad (1.34)\ldots\ldots$$

$(\nu = 1, 2, \ldots, l;\ j = 0, 1, 2, \ldots s)$.

We introduce the additional kinematic equations equal to the equations of the constraints.

$$e_{l+k} = F_k = \sum_{i=1}^{s} \alpha_{ki} \dot{q}_i + \alpha_{ko}. \qquad (1.34)\ldots\ldots$$

Suppose the relations (1.33) and (1.34) are independent of each other, then the system of s equations (1.33) and (1.34) will be solved in respect of $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s$. We get

$$\dot{q}_i = \sum_{\nu=1}^{l} \beta_{i\nu} e_\nu + \sum_{k=1}^{r} \beta_{i, l+k} e_{l+k} + \beta_{io}, \qquad (1.35)\ldots\ldots$$

where the coefficients β_{io}, β_{in} $(i, n = 1, 2, \ldots, s)$ are functions of $q_1, q_2, \ldots, q_s$ and t. Putting $e_{l+k} = 0$ $(k = 1, 2, \ldots, r)$, in (1.35) gives relations (1.16). However, this change is not made during the process of derivation, but only in the finally accomplished result. We obtain:

$$e_{l+k} = 0.$$

Since

$$\delta\pi_{l+k} = \sum_{i=1}^{s} \frac{\partial F_k}{\partial q_i}\,\delta q_i = 0,$$

variation in Lagrangian coordinates is given by

$$\delta q_i = \sum_{\nu=1}^{l} \frac{\partial \dot{q}_i}{\partial e_\nu}\,\delta\pi_\nu \ .$$

Therefore, equations (1.19) will be useful in the determination of kinematic response and their modifications viz. Chaplingin equations which will be written in the form

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial e_\nu} - \frac{\partial \tilde{T}}{\partial \pi_\nu} - \sum_{i=1}^{s} \frac{\partial T}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial \dot{q}_i}{\partial e_\nu} - \frac{\partial \dot{\gamma}_i}{\partial \pi_\nu}\right) =$$

$$= E_\nu + G_\nu \ , \ e_{l+k} = 0. \qquad (1.36).....$$

Here $\tilde{T}$ denotes the kinetic energy expressed in terms of original and additional kinematic responses with the help of (1.35) i.e.

$$\tilde{T} = T\left[q_i,\ q_i(e\ ,\ e_{l+k},\ q_n),\ t\right]$$

$(i, n = 1, 2, \ldots, s;\quad = 1, 2, \ldots, l;\ k = 1, 2, \ldots, r,\ s = l + r).$

Note that

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{n=1}^{s} \frac{\partial \tilde{T}}{\partial e_n} \frac{\partial e_n}{\partial \dot{q}_i} .$$

Therefore, (1.36) can be written as

$$\frac{d}{dt} \frac{\partial \tilde{T}}{\partial e_\nu} - \frac{\partial \tilde{T}}{\partial \pi_\nu} + \sum_{n=1}^{s} \frac{T}{e_n} W_\nu^n = M + G , \quad e_{l+k} = 0, \qquad (1.37).....$$

where the coefficients

$$W_\nu^n = - \sum_{i=1}^{s} \frac{\partial e_n}{\partial \dot{q}_i} \left(\frac{d}{dt} \frac{\partial \dot{q}_i}{\partial e_\nu} - \frac{\partial q_i}{\partial \pi_\nu} \right) \qquad (1.38).....$$

represent the kinematic values. These values depend only upon the selection of the coordinates and not on the dynamic characteristic of the system (masses, movements of the inertia etc.).

The coefficients W_ν^n ($\nu = 1, 2, \ldots, l$; $n = 1, 2, \ldots, s$) are called coefficients of the first type. Equations (1.37) are called equations in the mechanics of variable masses. The difference between check the ramo **Vorontz-Hamel** equations (1.37) and **Claligin** equations (1.26) consists in the fact that the former equations are can be completely solved. They do not contain the

terms $\frac{\partial T}{\partial \dot{q}_i}$ $(i = 1, 2, \ldots, s)$, which appears in terms of $\frac{\partial T}{\partial \dot{\sigma}_n}$ $(n = 1, 2, \ldots, s)$. It became possible only because of the introduction of the additional kinematic characteristics equal to the equations of the constraints.

In view of the relations

$$\sum_{i=1}^{s} \frac{\partial \sigma_n}{\partial \dot{q}_i} \frac{d}{dt} \frac{d\dot{q}_i}{d\sigma} = - \sum_{i=1}^{s} \frac{\partial \dot{q}_i}{\partial \sigma_\nu} \frac{d}{dt} \frac{\partial \sigma_n}{\partial \dot{q}_i},$$

$$\sum_{i=1}^{s} \frac{\partial \sigma_n}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \pi_\nu} = \sum_{i,i=1}^{s} \frac{\partial \sigma_n}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \sigma_\nu}$$

and the equation

$$\sum_{i=1}^{s} \frac{\partial \sigma_n}{\partial q_i} \frac{\partial \dot{q}_i}{\partial \dot{q}_j} + \frac{\partial \sigma_n}{\partial q_j} = 0,$$

which holds good, we obtain a final expression for the Voronets-Hamilya coefficients of the first type

$$W_\nu^n = \sum_{i=1}^{s} \frac{\partial \dot{q}_i}{\partial \sigma_\nu} \left(\frac{d}{dt} \frac{\partial \sigma_n}{\partial \dot{q}_i} - \frac{\partial \sigma_n}{\partial q_i} \right) \qquad (1.39).....$$

Let the relations (1.33) and the constraint equations and consequently the relations (1.35) be steady and homogeneous, then

$$W_v^n = \sum_{i=1}^{s} \beta_{i,} \frac{d}{dt} \alpha_{ni} - \sum_{j=1}^{s} \frac{\partial \alpha_{ni}}{\partial q_i} \dot{q}_j \quad .$$

We changed the number in coefficients α_{ni}, by denoting $\alpha_{r+s,i}$ by α_{si} and α_{ki} by $\alpha_{l+k,i}$. We thus get

$$W_v^n = \sum_{i,j=1}^{s} \left(\frac{\partial \alpha_{ni}}{\partial q_j} - \frac{\partial \alpha_{nj}}{\partial q_i} \right) \beta_{i,} \dot{q}_j$$

or on the basis of the homogeneity(uniformity) of the kinematic characteristics (1.33) in the present case

$$W_v^n = \sum_{\gamma=1}^{l} e_\gamma W_{\gamma,}^n \quad , \qquad (1.40).....$$

where

$$W_{\gamma,}^n = \sum_{i,j=1}^{s} \left(\frac{\partial \alpha_{ni}}{\partial q_j} - \frac{\partial \alpha_{nj}}{\partial q_i} \right) \beta_{j\gamma} \beta_{i,}$$

The coefficients γ which depend only upon the Largangian coordinates and time are called veronsa-Hannily coefficients of the second type.

Equations (1.37) take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{e}} - \frac{\partial T}{\partial} + \sum_{m=1}^{s}\sum_{j=1}^{l} \frac{\partial T}{\partial e_m} \gamma \; e_j = E + G \; , \; e_{l=k} = 0.$$

(1.41).....

For the holonamic systems with constant masses equations (1.41) were obtained by P.V. Vorontz in the year 1901 (9). In the same year he published another work (10) containing all the necessary details on how to extend the above equations to the non-holonamic systems. In the year 1904, Hamel obtained with constant masses (110).

It is for this reason that equations (1.41) are named after varpusa and Hannilya.

2. Equations of Motion using special derivatives.

1. Equations with undetermined multipliers and explicit calculation of reaction of the non-holonomic constraints. If we repeat the computations given in § 1, paragraph one, for the general equation of the mechanics of a variable mass in the form of (8.4) of Chapter II, we get the following equations with

$$\frac{d^*}{dt}\frac{\partial^* T}{\partial \dot{q}_i} - \frac{\partial^* T}{\partial q_i} = Q_i + V_i + \sum_{k=1}^{r} \lambda_k \frac{\partial F_k}{\partial \dot{q}_i}, (i = 1, 2, \ldots, s).$$

(2.1).....

It is easy to see that the equations with undetermined multipliers (1.6) can be written in the form (2.1). However, the independent derivation of (2.1) from the general equations of the mechanics enables us to see that these equations are more useful to treat the general functional dependence of mass than these of equations (1.6).

This form of equations (2.1) is useful for solving concrete problems. Therefore, it is convenient to write undetermined multipliers in terms of derivatives used in equations (2.1). The direct transformation of expression (1.15) presents a difficulty because in that case we have to replace derivatives in the formulae which determine the elements $A_{k,\mu}$. It is also to be noted that the result (1.15) does not always hold because masses of the points of the system may depend besides the time, the legranges coordinates and the velocities, upon certain additional variables like paths traversed by individual points of the mechanical system.

Let us briefly compute the end of p. 1 Section 1 for the present case.

Now examine the generalized impulses $p'_i = \frac{\partial^* T}{\partial \dot{q}_i}$.

From this we get

$$\dot{q}_i = \dot{q}_i(q_m, p'_m, t, m_j). \qquad (2.2).....$$

Take

$$F'_k = F_k(q_i, \dot{q}_i(q_m, p'_m, t, m_j), t). \qquad (2.3).....$$

We have

$$\frac{d}{dt} F'_k = \frac{d^*}{dt} F'_k + \sum_j \left[\sum_{i=1}^{s} \frac{\partial F_k}{\partial q_i} \left(\sum_{m=1}^{s} \frac{\partial q_i}{\partial p'_m} \frac{\partial p'_m}{\partial m_j} + \frac{\partial q_i}{\partial m_j} \right) \right] \dot{m}_j = 0.$$

The expressions standing in the brackets are equal to zero because

The expressions in brackets are equal to zero, because they are partial derivatives of m_j, similar to the transformed equations (2.2), if we take $p_i = \frac{\partial^* T}{\partial qi}$.

Write in detail $\frac{d^*}{dt} F'_k = 0$ and put it in the values of $\frac{d^*}{dt} p'_i$, as obtained from equations (2.1), we get the following result for undetermined multipliers

$$\lambda_k = - \sum_{\mu=1}^{r} B'_{k\mu} \frac{DF'_\mu}{Dt}. \qquad (2.4).....$$

In this $B'_{k\mu}$ are the elements of the matrix which is inverse te the matrix whose elements are

$$A'_{k\mu} = \sum_{i=1}^{s} \frac{\partial F'_k}{\partial P'_i} \frac{\partial F_\mu}{\partial q_i} . \qquad (2.5).....$$

Denote

$$\frac{DF'_\mu}{Dt} = \sum_{i=1}^{s} \frac{\partial F'_\mu}{\partial P'_i} \left(\frac{\partial T}{\partial q'_i} + Q_i + \Psi_i \right) + \sum_{i=1}^{s} \frac{\partial F'_\mu}{\partial q_i} \dot{q}_i +$$

$$+ \frac{\partial F'_\mu}{\partial t} . \qquad (2.6).....$$

$\frac{DF'_\mu}{Dt}$ represent the derivatives in respect of time of equation (2.3), which are computed when masses are fixed and the effect of the constraint reaction is not taken into account. The formulae (2.4) was worked out without any restriction on the functional dependence of mass of points.

Put $\Psi_i = 0$, in (2.4) and it gives undetermined multipliers in the non-holonomic mechanics of the constraint mass [60] . Therefore,

$$\Lambda^i = - \sum_{k,\mu=1}^{r} B'_{k\mu} \sum_{m=1}^{s} \frac{\partial F'_\mu}{\partial P'_m} \Psi_m \frac{\partial F_k}{\partial q_i} \qquad (2.7).....$$

represents that part of i-th generalized reaction of the non-holonomic constraints, which is caused by the process of change of mass.

2. Equation of Motion of Chapligin and Vorontz-Hamel:

In the present investigation, starting of the general equation of mechanics differs from the earlier used general mechanics equation in the particular derivatives and the availability of the generalized reaction forces in places of surplus forces.

In conformity with the derivation of equation in the non-holonomic coordinates, chaplinga equations can be put in agreed notation as

$$\frac{dt^{\circ}}{dt}\frac{\partial^{\circ}T}{\partial \dot{e}_{\nu}} - \frac{\partial^{\circ}T}{\partial \pi_{\nu}} - \sum_{i=1}^{s} \frac{\partial^{\circ}T}{\partial \dot{q}_i} L_{e\nu}(\dot{q}_i) = E_{\nu} + \Phi_{\nu} ,$$

(2.8).....

Where $\Phi = \sum_{i=1}^{s} \Psi_i \frac{\partial \dot{q}_i}{\partial \dot{e}_{\nu}}$ is generalized creation force in respect of non-holonomic coordinate.

For a chaplinga system with variable masses let us take lagrangian coordinates in place of the non-holonomics, then equations (2.8) are transferred to

$$\frac{d^{\circ}}{dt}\frac{\partial^{\circ}T}{\partial \dot{q}_{\nu}} - \frac{\partial^{\circ}T}{\partial q_{\nu}} - \sum_{k=1}^{r} \frac{\partial^{\circ}T}{\partial \dot{q}_{l+k}} \left(\frac{d}{dt}\frac{\partial \dot{q}_{l+k}}{\partial \dot{q}_{\nu}} - \frac{\partial \dot{q}_{l+k}}{\partial q_{\nu}} \right) = Q_{\nu} + \Psi_{\nu} .$$

(2.9).....

Voreusa-Hannilya equations become

$$\frac{\partial \dot{a}^*}{dt}\frac{\partial^* T}{\partial e_\nu} - \frac{\partial^* \tilde{T}}{\partial \pi_\nu} + \sum_{n=1}^{s} \frac{\partial^* \tilde{T}}{\partial e_n} W_\nu^n = \Sigma + \Theta_\nu, \; e_{l+k} = 0.$$

(2.10).....

From the way the equations (2.8) - (2.10) are written, a rule may be formulated. In order to construct these equations when the motion of a certain non-holonomic system is studied, we write the equations for a corresponding problem with constant masses.

How to apply equations (2.8) - (2.10) to the non-linear and non-holonomic system with variable masses has been discussed in authors work (64).

3. Gauss's Principle.

1. Foundation of the Principle. Consider Gauss's function

$$G = \sum_j m_j \left(\frac{\bar{dv}_j}{dt} - \frac{F_j}{m_j} - \frac{\bar{R}_j}{m_j} \right)^2 . \qquad (3.1).....$$

In this summation is taken over all the points of the system, $\frac{\bar{dv}_j}{dt}$ - is accelcertion of jth point on an arbitrary kinematic trajectory whose coordinates and velocities at a given moment of time coincide with the coordinates and velocities of the actual trajectory. The masses m_j and action forces $\bar{F}_j$ on varying bunching trajectory are the same as on actual trajectory (reactions are excluded because the constraints are ideal). The reaction forces $\bar{R}_j$ which depend upon the acceleration do not vary when (on

transition to bunching tracejecteries). We change over to the varying trajecteries.

Let a mechanical system be subjected to the linear non-holenomic constraints which can be written in the Cartesian coerdinates.

$$f_k = \sum_{\gamma} a_{k\gamma} \dot{x}_{\gamma} + a_{ke} = 0, \; a_{k\gamma} = a_{k\gamma}(x_p, t), \qquad (3.2).....$$

$$a_{ke} = a_{ke}(x_p, t) \; (k = 1, 2, ..., r).$$

Indices γ and p assume values in whole number for all Cartesian coerdinates of the mechanical system. For the convenience of writing we denote the cartesian coerdinates with reference to there mutually perpendicular axes by the letter x with the corresponding indices.

The variations satisfy the following conditions

$$\sum_{\gamma} a_{k\gamma} \delta x_{\gamma} = 0. \qquad (3.3).....$$

The differentiating equations (3.2) along the trajectory of the mechanical system

$$\sum_{\gamma} a_{k\gamma} \ddot{x}_{\gamma} + \Gamma_{\gamma}(x_p, \dot{x}_p, t) = 0. \qquad (3.4).....$$

where Γ_{γ} is a function of the indicated arguments.

Since the bunching and the actual trajecteries have the same velocities, coordinates and time, the subtraction of equations (3.4) for the two trajecties, therefore, gives

$$\sum_{\gamma} a_{k\gamma} (\ddot{x}_{\gamma} - \ddot{x}^{\bullet}_{\gamma}) = 0, \qquad (3.5).....$$

where $\ddot{x}_{\gamma}$ - is the component of the acceleration of j-th point is the component of the acceleration of $\ddot{x}^{\bullet}_{j}$ point on the actual traijectory. Indices j of the point and of the cartesian acceleration are connected by the relation

$$\gamma = 3(j - 1) + q . \qquad (3.6).....$$

q assumes the values 1,2 or 3 and indicates the cartesian axis on which the projection takes place.

It is evident from the relations (3.3) and (3.5) that we may take values proportional to the difference as virtual displacements δx_{γ} in the general equation of mechanics. Denoted by $\ddot{x}_{\gamma} - \ddot{x}^{\bullet}_{\gamma}$, the projection of reaction force applied to j^{th} point on q – axis and by X_{γ} the coresponding projection of the force $\bar{F}_j$, the general equation of the mechanics in carterian coordinates is written as

$$\sum_{\gamma} (m_{\gamma} \ddot{x}^{\bullet}_{\gamma} - X_{\gamma} - R_{\gamma}) (\ddot{x} - \ddot{x}^{\bullet}_{\gamma}) = 0. \qquad (3.7).....$$

The double equation (3.7) may be presented as

$$G = G^{\bullet} + \sum_{\gamma} m (\ddot{x} - \ddot{x}^{\bullet}_{\gamma})^2. \qquad (3.8).....$$

As the second component on the right hand side of (3.8) is positive for any bunching trajectory, the function G, therefore, attains a minimum on the actual trajectory.

The embodies Gauss's principle for a mechanical system with variable masses, which can be formulated as under. The Gauss's function G (described in (3.1) attains the minimum on the actual trajectory of the system with variable masses, in comparison with similar (congruent) trajecteries which at a given time have the same coordinates, velocities and reaction forces as the actual trajectory.

2. Equations of Gauss and Apple.

Gauss's equations will provide the necessary condition of minimum G.

$$\frac{\partial' G}{\partial' \ddot{e}_{\nu}} = 0 \quad (\nu = 1, 2, ..., l; \; l = s - r), \qquad (3.9).....$$

where prime in the derivative means that while computing the derivative in respect of $\ddot{e}_{\nu}$ reaction forces are not differentiated.

Gauss's function can be expressed in the form

$$G = 2A - 2 \sum_{\nu=1}^{l} (E_\nu + \Theta_\nu)\, \dot{e}_\nu + \Gamma(q_i, e_\nu, t),$$

where

$$A = \frac{1}{2} \sum_{\gamma} m_\gamma \ddot{x}_\gamma^2$$

function in the indicated variables. Apple's equations for the non-holonomic mechanics follow from equations (3.9)

$$\frac{\partial A}{\partial \dot{e}_\nu} = E_\nu + \Theta_\nu \qquad (3.10).....$$

4. Motion of the Non-holonomic System with Constraints depending upon the process of change of Mass.

1. Equation of Motion. Let the motion of a mechanical system be subjected to the holonomic as well as non-holonomic constraints which may depend upon the mass of point. If some holonomic constraints depend upon the mass only and may be expressed as functions of coordinates and time, then on account of variation in mass they will remain holonomic. A set of s lagrangian coordinates q_i may identically satisfy all such constants. We should also take into consideration the non-

holonomic constraints which are obtained from the original non-holonomic constraints altered by the variation in mass and also from those original holonomic constraints which are reduced to non-holonomic as a consequence of the dependence of mass upon velocity. Let the non-holonomic thus obtained constraints satisfy a set of the kinematic characteristics e_ν, $\nu = 1, 2, \ldots, l$; $l = s-r$, where r is the number of finally obtained non-holonomic constraints.

The general equation of the mechanics written in terms of derivatives with stars leads to s equations with undetermined m multipliers

$$\frac{d^*}{dt}\frac{\partial^* T_D}{\partial \dot{q}_i} - \frac{\partial^* T_D}{\partial q_i} = Q_i' + \Psi_i' + \Lambda_i', \quad \Lambda_i = \sum_{k=1}^{r} \lambda_k \frac{\partial F_k'}{\partial q_i}, \tag{4.1}$$

where λ_k are undetermined multipliers for the evaluation of which an explicit expression may be found. $F_k'(q_i, \dot{q}_i, t) = 0$ are the finally obtained equations of the non-holonomic constraints. As before, $T_D(k_j, q_i, \dot{q}_i, t)$ denotes the kinetic energy of the system calculated on the basis of kinematic change in the process of change of mass, i.e.,

$$T_D = \frac{1}{2} \sum_j m_j v_j^2 \, (q_i, \dot{q}_i, t).$$

In this $v_j(q_i, q_i, t)$ account for the new holonomic constraints obtained after taking into consideration the process of change of mass.

As shown in sections 1 and 2 the chaplinge equations and the veronsa-Hamilya equations are respectively written in the form

$$\frac{d^*}{dt}\frac{\partial^* T_D}{\partial e_\nu} - \frac{\partial^* \tilde{T}_D}{\partial \pi_\nu} - \sum_{i=1}^{s} \frac{\partial^* T_D}{\partial q_i} L'_{e\nu}(q_i) = E'_\nu + \Theta'_\nu, \qquad (4.2)....$$

$$\frac{d^*}{dt}\frac{\partial^* \tilde{T}_D}{\partial e_\nu} - \frac{\partial^* \tilde{T}_D}{\partial \pi_\nu} + \sum_{n=1}^{s} \frac{\partial^* \tilde{T}_D}{\partial e_n} W'^{n}_{\nu} = E'_\nu + \Theta'_\nu, \qquad (4.3)....$$

where $\tilde{T}_D$ represent the kinetic energy T_D, in which q_i are expressed in terms of e_ν , q_i, t, while m_j are fixed. The derivatives are computed keeping mass of the points fixed in functions T_D and $\tilde{T}_D$. Sign (prime) means that corresponding values are determined on account of finally obtained equations of the holonomic and the non-holonomic constraints.

In chaplinga systems if the independent lagrangian velocities are taken as the kinematic characteristics e_ν then chaplinga equations (4.2) may be written in the form

$$\frac{d^*}{dt}\frac{\partial^*\tilde{T}_D}{\partial \dot{q}_\nu} - \frac{\partial^*\tilde{T}_D}{\partial q_\nu} - \sum_{k=1}^{r} \frac{\partial^* T_D}{\partial \dot{q}_{l+k}} L_{q_\nu}(\dot{q}_{l+k}) = Q'_\nu + \Psi'_\nu +$$

$$+ \sum_{k=1}^{r} (Q'_{l+k} + \Psi'_{l+k}) \frac{\partial \dot{q}_{l+k}}{\partial \dot{q}_\nu}, \qquad (4.4).....$$

where chaplinga coefficients are equal to

$$L_{q_\nu}(\dot{q}_{l+k}) = \frac{d}{dt}\frac{\partial \dot{q}_{l+k}}{\partial \dot{q}_\nu} - \frac{\partial \dot{q}_{l+k}}{\partial q_\nu} \qquad (4.5).....$$

Equations (4.1) - (4.4) can be written in terms of regular partial derivatives.

The application of equations (4.1) - (4.4) to the non-carrier non-holonomic systems with variable masses has been discussed in the authors work (64).

2. Examples.

A cart (mentioned earlier in §3 of the last chapter) will serve as an example of the non-holonomic system with constraints depending upon the process of change of mass if the plane of the front wheel is inclined to the plane of the rear wheels.

We will examine, in detail, simpler example of the rolling without side-ways motion on the horizontal plane of the wheel made up of absolutely flexible band of thickness h and the linear density γ. The example is interesting is the sense that for a partly (special) non-holonomic system the chaplinga

equations degenerate into Lagrangian equations.

The position of the wheel will be determined by cartesian coordinates of the center x, y, z, and Euler's two angles φ, ψ. Denoting by φ the angle of rotation of wheel and by the angle between the axis of wheel and x-axis, we get

$$T = \frac{m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{2} + \frac{A\dot{\psi}^2 + C\dot{\varphi}^2}{2},$$

where m is the mass of wheel. A and C are corresponding moments of inertia. Coordinate z is equal to the radius.

By analogy to the problem given in para 3, § 3 of Chapter III, we find

$$z^2 = \frac{h}{\pi}(l - s),$$

where l is the length of band forming the wheel initially. s is the distance of the point contact. Since $s = z\varphi$, differentiating the last equation we get

$$\dot{z} = -\frac{1}{2}\frac{h}{\pi}\dot{\varphi}, \quad z = r - \frac{1}{2}\frac{h}{\pi}\varphi,$$

where r is initial radius of wheel, $r = \sqrt{\frac{hl}{\pi}}$

Kinetic energy T_D is given as

$$T_D = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{A\dot{\psi}^2 + \left(C + \frac{mh^2}{4\pi^2}\right)\dot{\varphi}^2}{2} .$$

The motion of the system satisfies the non-holonomic constraints

$$\dot{x} = \left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\varphi}\cos\psi ,$$

$$\dot{y} = \left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\varphi}\sin\psi .$$

Such a system is called chaplinga system.

Chaplinga coefficients have values

$$L_\varphi(\dot{x}) = -\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\psi}\sin\psi,\ L_\varphi(\dot{y}) = \left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\psi}\cos\psi ,$$

$$L_\psi(\dot{x}) = \left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\varphi}\sin\psi,\ L_\psi(\dot{y}) = -\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\varphi}\cos\psi .$$

Since chaplinga coefficients are different from zero, so the above constraints can not be integrated but it is easy to see that

$$mxL_\varphi(\dot{x}) + myL_\varphi(\dot{y}) = 0,$$

$$mxL_\psi(\dot{x}) + myL_\psi(\dot{y}) = 0.$$

Therefore, chaplinga equations (3.4) degenerate into Lagrange's equations.

The transformed kinetic energy is written as

$$\tilde{T}_D = \frac{A\dot{\psi}^2}{2} + \frac{C + \frac{mh^2}{4\pi^2} + m\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)^2}{2}\dot{\varphi}^2 .$$

Generalized reaction faces are equal to zero; further $mg\,\delta z = \frac{mgh}{2\pi}\delta\varphi$, $Q'_\psi = \frac{mgh}{2\pi}$. This gives the following equation of motion

$$A\ddot{\psi} = 0,$$

$$\left[C + \frac{mh^2}{4\pi^2} + m\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)^2\right]\ddot{\varphi} - \frac{1}{2}\frac{mh}{\pi}\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)\dot{\varphi}^2 = \frac{mgh}{2\pi} .$$

From first equation, we have $\dot{\psi} = \dot{\psi}_0$. In second equation, put

$$m = \frac{\pi\gamma}{h}\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)^2 , \quad C = \frac{mz^2}{2} = \frac{\pi\gamma}{2h}\left(r - \frac{1}{2}\frac{h}{\pi}\varphi\right)^4 .$$

The substitution

$$\varphi = \frac{2\pi}{h}\left(r - \sqrt{\frac{h(1-s)}{\pi}}\right)$$

reduces the obtained equation to the form (3.4) of Chapter III in the variables s^2 and s.

Problem 2. Let us study rolling due to inertia along the horizontal plane of homogeneous sphere (ball). The decrease in mass is proportional to the surface area of the sphere

$$\frac{dm}{dt} = -4\pi\varkappa z^2,$$

where $\varkappa$ = constant; z - radius of sphere. Mass of sphere $m = \frac{4}{3}\pi\gamma z^3$, where γ is constant density of sphere. Denoting the initial radius by , we get

$$z = r - \frac{\varkappa}{\gamma} t .$$

Kinetic energy of sphere is (61).

$$T_D = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \frac{\varkappa^2}{\gamma^2}\right) + \frac{I}{2}\left(\dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta + \dot{\theta}^2\right),$$

where $I = \frac{2}{5} mz^2$ is the moment of inertia of sphere about the diameter. x, y, z are horizental coordinates of the center of sphere φ, ψ, θ are Eulerian angles.
Lagrangian coordinates and velocities must satisfy the equations of non-holonomic constraints.

$$\dot{x} - \left(\dot{r} - \frac{\varkappa}{\gamma}\right)\left(-\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi\right) = 0,$$

$$\dot{y} + \left(\dot{r} - \frac{\varkappa}{\gamma}\right)\left(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right) = 0.$$

Thus there emerges chaplinga system of which the transformed kinetic energy is written in the form

$$T_D = \frac{m\left(r - \frac{x'}{\gamma} t\right)^2}{2} \left(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{m x'^2}{2\gamma^2} +$$

$$+ \frac{I}{2}\left(\dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta + \dot{\theta}^2\right) .$$

We find equations of the type (4.4) in variables φ, ψ, θ. Because generalized forces are equal to zero, therefore, we have

$$m\frac{d}{dt}\left[\left(r - \frac{x'}{\gamma} t\right)^2 \dot{\varphi}\sin^2\theta\right] + I\frac{d}{dt}\left(\dot{\varphi} + \dot{\psi}\cos\theta\right) -$$

$$- m\left(r - \frac{x'}{\gamma} t\right)^2 \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\dot{\theta}\sin\theta + \frac{x'm}{\gamma}\left(r - \frac{x'}{\gamma} t\right)\dot{\varphi}\sin^2\theta = 0,$$

$$I\frac{d}{dt}\left(\dot{\psi} + \dot{\varphi}\cos\theta\right) = 0,$$

$$m\frac{d}{dt}\left[\left(r - \frac{x'}{\gamma} t\right)^2 \dot{\theta}\right] + I\ddot{\theta} - m\left(r - \frac{x'}{\gamma} t\right)^2 \dot{\varphi}^2 \sin\theta\cos\theta +$$

$$+ I\dot{\varphi}\dot{\psi}\sin\theta + m\left(r - \frac{x'}{\gamma} t\right)^2 \left(\dot{\psi} + \dot{\varphi}\cos\theta\right)\dot{\varphi}\sin\theta + \frac{x'm}{\gamma}\left(r - \frac{x'}{\gamma} t\right)\dot{\theta} = 0.$$

After making certain transformations, we have

$$\frac{d}{dt}\left(\dot{\varphi}\sin^2\theta\right) + \frac{2}{5}\frac{d}{dt}\left(\dot{\varphi} + \dot{\psi}\cos\theta\right) - m\dot{\theta}\sin\theta = \frac{x'}{\gamma\left(r - \frac{x'}{\gamma} t\right)}\dot{\varphi}\sin^2\theta ,$$

$$+ \quad \cos \quad = n = \text{const},$$

$$\frac{7}{5}\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta + \frac{2}{5}\dot{\psi}\dot{\varphi}\sin\theta + n\dot{\varphi}\sin\theta = \frac{\chi'}{\gamma\left(r - \frac{\chi}{\gamma}t\right)}\dot{r}.$$

These equations may be written in the form

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot{\varphi}} - \frac{\partial T'}{\partial \varphi} = \frac{\chi'}{\gamma\left(r - \frac{\chi}{\gamma}t\right)}\dot{\varphi}\sin^2\theta, \qquad (4.6).....$$

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot{\psi}} - \frac{\partial T'}{\partial \psi} = 0, \qquad (4.7).....$$

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot{\theta}} - \frac{\partial T'}{\partial \theta} = \frac{\chi'}{\gamma\left(r - \frac{\chi}{\gamma}t\right)}\dot{\theta} \qquad (4.8).....$$

where

$$T' = \frac{7}{10}\left(\dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta + \dot{\theta}^2\right).$$

Now consider the case when the ball (sphere) is gradually rested. Suppose $\chi' = 0$, we get a system of equations which can be conveniently integrated by method of Jacobi (61). Total integral of Hamilton-Jacobi equations

$$\frac{\partial V}{\partial t} + \frac{5}{14\sin^2\theta}\left[\left(\frac{\partial V}{\partial \varphi}\right)^2 + \left(\frac{\partial V}{\partial \psi}\right)^2 - 2\frac{\partial V}{\partial \varphi}\frac{\partial V}{\partial \psi}\cos\theta + \right.$$

$$\left. + \left(\frac{\partial V}{\partial \theta}\right)^2 \sin^2\theta\right] = 0$$

is equal to

$$V = a_1\varphi + b_2\psi + f(\theta, a_1, a_2, a_3) - \frac{5}{14}a_3^2 t,$$

where

$$f(\theta, a_1, a_2, a_3) = \int\sqrt{a_3^2\sin^2\theta + 2a_1a_2\cos\theta - (a_1^2 + a_2^2)}\,\frac{d\theta}{\sin\theta},$$

a_i = const.

From here we obtain the partial integrals

$$\varphi + \int \frac{a_2\cos\theta - a_1}{\sqrt{a_3^2\sin^2\theta + 2a_1a_2\cos\theta - a_1^2 - a_2^2}}\,\frac{d\theta}{\sin\theta} = b_1 = \text{const}, \quad (4.9)\ldots$$

$$\psi + \int \frac{a_1\cos\theta - a_2}{\sqrt{a_3^2\sin^2\theta + 2a_1a_2\cos\theta - a_1^2 - a_2^2}}\,\frac{d\theta}{\sin\theta} = b_2 = \text{const}, \quad (4.10)..$$

$$-\frac{5}{7}a_3 t + \frac{a_3\sin\theta\,d\theta}{\sqrt{a_3^2\sin^2\theta + 2a_1a_2\cos\theta - a_1^2 - a_2^2}} = b_3 = \text{const}. \quad (4.11)..$$

(4.11) gives the value of θ,

$$\theta = \arccos\frac{1}{a_3^2}\left[a_1a_2 - \sqrt{(a_3^2 - a_1^2)(a_3^2 - a_2^2)}\,\cos\left(\frac{5}{7}a_3 t + b_3\right)\right].$$

(4.12).....

Differentiating (4.11) with respect to time, we get

$$\dot{\theta}\sin\theta = \frac{5}{7}\sqrt{a_3^2\sin^2\theta + 2a_1a_2\cos\varphi - a_1^2 - a_2^2}\ ,$$

Therefore, (4.9) - (4.10) are transformed to

$$\varphi + \frac{5}{7}\int \frac{a_2\cos\theta - a_1}{\sin^2\theta}\,dt = b_1, \qquad (4.13).....$$

$$\varphi + \frac{5}{7}\int \frac{a_1\cos\theta - a_2}{\sin^2\theta}\,dt = b_2. \qquad (4.14).....$$

Taking into consideration formulae (4.12) - (4.14) and denoting

$$a^2 = \sqrt{(a_3^2 - a_1^2)(a_3^2 - a_2^2)},\ \tau = \operatorname{tg}\frac{1}{2}\left(\frac{5}{7}a_3t + b_3\right),$$

the following expressions for angles are found

$$\varphi = b_1 + \frac{a_3\sqrt{(a_3^2 + a^2 + a_1a_2)(a_3^2 - a^2 - a_1a_2)}}{a_1^2(a_3^2 - a_2^2) + a_2^2(a_3^2 - a_1^2)}\ \times$$

$$\times\left[(a_1 - a_2)\sqrt{\frac{a_3^2 + a^2 + a_1a_2}{a_3^2 + a^2 - a_1a_2}}\operatorname{arctg}\tau\sqrt{\frac{a_3^2 - a^2 - a_1a_2}{a_3^2 + a^2 - a_1a_2}} + \right.$$

$$+(a_1 + a_2)\sqrt{\frac{a_3^2 - a^2 - a_1a_2}{a_3^2 - a^2 + a_1a_2}}\,\mathrm{arctg}\sqrt{\frac{a_3^2 + a^2 + a_1a_2}{a_3^2 - a^2 + a_1a_2}}, \qquad (4.15).....$$

$$\varphi = b_2 + \frac{a_3\sqrt{(a_3^2 + a^2 + a_1a_2)(a_3^2 - a^2 - a_1a_2)}}{a_1^2(a_3^2 - a_2^2) + a_2^2(a_3^2 - a_1^2)} \times$$

$$\times\left[-(a_1 - a_2)\sqrt{\frac{a_3^2 + a^2 + a_1a_2}{a_3^2 + a^2 - a_1a_2}}\,\mathrm{arctg}\sqrt{\frac{a_3^2 - a^2 - a_1a_2}{a_3^2 + a^2 - a_1a_2}} + \right.$$

$$\left. + (a_1 + a_2)\sqrt{\frac{a_3^2 - a^2 - a_1a_2}{a_3^2 - a^2 + a_1a_2}}\,\mathrm{arctg}\sqrt{\frac{a_3^2 + a^2 + a_1a_2}{a_3^2 - a^2 + a_1a_2}}\right].$$

(4.16).....

We choose the constants a_i and b_i ($i = 1, 2, 3,$) in such a way that the functions (4.12), (4.15), (4.16) and their derivatives are satisfied by the initial conditions of the ball with variable mass. These constants are denoted by zero. Under slow heating of the ball the obtained functions will be approximately equal to the angles sought. For greater accuracy, we calculate right side parts of equations (4.6) - (4.8) and restrict them to terms containing small multiplier and to the first order using results

(4.12) and (4.15). We get

$$f_1(a_i^\circ, b_i^\circ, t) = \frac{\alpha}{\gamma r}\dot{\varphi}\sin^2\theta\,,\ 0,\ f_2(a_i^\circ, b_i^\circ, t) = \frac{\alpha}{\gamma r}\dot{\theta}\,, \qquad (4.17)\ldots$$

The system of equations thus obtained is called a canonical system with Hamilton function $H = T' + f_1\varphi + f_2\theta$. Therefore, in accordance with perturbance theory widely used in celestial mechanics (20), solution of above equations is given by results (4.12), (4.15) and (4.16) in which a_i, b_i are determined by canonical equations

$$-\frac{da_i}{dt} = f_1\frac{\partial\varphi}{\partial b_i} + f_2\frac{\partial\theta}{\partial b_i}\,,$$

$$\frac{db_i}{dt} = f_1\frac{\partial\varphi}{\partial a_i} + f_2\frac{\partial\theta}{\partial a_i} \qquad (4.18)\ldots\ldots$$

and have initial values a_i°, b_i°.

For example, consider rolling of ball with initial conditions

$$\varphi_0 = \psi_0 = \theta_0 = 0,\ \dot{\varphi}_0 = \dot{\psi}_0 = 0,\ \dot{\theta}_0 = \omega \qquad (4.19)\ldots\ldots$$

On account of (4.12) - (4.17) and (4.19) we get

$$a_1^\circ = a_2^\circ = 0,\ b_1^\circ = b_2^\circ = 0,\ a_3^\circ = \frac{2\omega}{5}\,,\ b_3^\circ = \pi\,,\ f_1 = 0,\ f_2 = \frac{\alpha\omega}{\gamma r}\,.$$

Equations (4.18) in the present case take the form

$$\frac{da_1}{dt} = 0, \quad \frac{da_2}{dt} = 0, \quad \frac{da_3}{dt} = \frac{\varkappa\omega}{\gamma r},$$

$$\frac{db_1}{dt} = 0, \quad \frac{db_2}{dt} = 0, \quad \frac{db_3}{dt} = -\frac{5}{7}\frac{\varkappa\omega}{\gamma r}t.$$

This gives

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = \frac{7\omega}{5} + \frac{\varkappa\omega}{\gamma r}t,$$

$$b_1 = 0, \quad b_2 = 0, \quad b_3 = \pi - \frac{5}{14}\frac{\varkappa\omega}{\gamma r}t^2.$$

Finally, from (4.12), (4.15) and (4.16), we find

$$\varphi = 0, \quad = 0,$$

$$\theta = \omega t\left(1 + \frac{5}{14}\frac{\varkappa}{\gamma r}t\right).$$

Consequently, the decrease in mass accelerates the rolling (rotation) of the ball. Horizontal velocity of the center of ball is equal to

$$\dot{\xi} = \beta\left(r - \frac{\varkappa}{\gamma}t\right).$$

Restricting to terms containing to the first order

$$\dot{\xi} = \omega r - \frac{2}{7}\frac{\varkappa\omega}{\gamma}t,$$

i.e. horizontal velocity of the center of ball diminishes.

CHAPTER V

SPECIAL MECHANICAL CHARACTERISTIC OF BODIES WITH VARIABLE MASSES.

1. Principal Vector of the Reaction Forces.

In the present panagraph we will obtain the results for the principal vector of cariolin forces and also for the principal vector of the remaining part of the reaction forces. There results in the case of a rocket were proposed (formulated) by F.R. Gantmakher and L.M. Levin in the year 1947.

1. Formula for the Principal Vector of the Cariolin Forces.

Principal vector for the carioliso forces may be written in the form

$$\bar{J} = -2\sum \bar{\omega} \times m\bar{u}_r,$$

where the summation is taken over all the particles $\bar{\omega}$ is angular velocity of the rotation of the body, $\bar{u}_r$ is relative velocity of certain particle, m is its mass differentiating the sum in respect of time we have

$$\bar{C} = \sum_{(t)} m\bar{r}, \qquad (1.1).....$$

which is computed for all particles present in the body, $\bar{r}$ is radius vector of a certain particle with reference to the fixed origin. In order to differentiate we will think of the sum (1.1), on the one hand, as the sum over geometrical points of

the body having variable mass and, on the other hand, as the sum over moving particles of the constant mass.

It may be noted that the composition of the particles forming the body of variable mass will vary. Therefore, the sum (1.1) will, at different times, be computed for different sets of particles, i.e., at time t it will be taken over by all the particles included in the composition of the body at the given time.

Differentiating according to the former of the two indicated modes, we have

$$\frac{d}{dt}\bar{c} = \frac{d}{dt}\iiint_{(\tau)} \bar{r}\rho\, d\tau = \iiint_{(\tau)} \bar{r}\frac{\partial \rho}{\partial t}\, d\tau + \iiint_{(\tau)} \bar{v}\rho\, d\tau , \qquad (1.2)....$$

where integration is performed over the volume of the body, while the domain of the integration is fixed, assuming that the density $\rho(x, y, z, t)$ for certain points is equal to zero. In (1.2), represent the velocity of an arbitrary geometrical point of the body.

Now considering the moving particles of the constant mass, we get

$$\frac{d}{dt}\bar{c} = \lim_{\Delta t \to 0} \frac{\bar{c}(t + \Delta t) - \bar{c}(t)}{\Delta t} . \qquad (1.3).....$$

Vector $\bar{c}$ at time $t + \Delta t$ is equal to

$$\bar{\mathcal{C}}(t+\Delta t) = \sum_{(t+\Delta t)} m(\bar{r}+\Delta\bar{r}) =$$

$$= \sum_{(t)} m(\bar{r}+\Delta\bar{r}) + \sum'_{(1)} \Delta\mu\,\bar{r} - \sum'_{(2)} \Delta\mu\,\bar{r},$$

where $\sum'_{(1)}$ is taken over all the particles entering into the body in interval $[t,\ t+\Delta t]$, and $\sum'_{(2)}$ is taken over all the particles objecting from the body with variable mass at the same time. $\Delta\mu$ denotes in the two cases the corresponding mass of particles.

We can write relation (1.3) in the form

$$\frac{d}{dt}\bar{\mathcal{C}} = \sum_{(t)} m\bar{u} + \lim_{\Delta t\to 0} \sum'_{(1)} \frac{\Delta\mu}{\Delta t}\bar{r} - \lim_{\Delta t\to 0} \sum'_{(2)} \frac{\Delta\mu}{\Delta t}\bar{r} \quad . \qquad (1.4)....$$

Here $\bar{u}$ is absolute velocity of particle of mass m. The first sum, on the right side of (1.4), is taken over all the particles composing the body at the time under consideration t, i.e., over the particles flowing in the body and also over the particles of the frame of the body, therefore, taking into consideration that

$$\bar{u} = \bar{v} + \bar{u}_p, \sum' m\bar{v} = \iiint_{(\tau)} \bar{v}\rho\, d\tau\ ,$$

and also the fact that for particles of frame $\bar{u}_r = 0$ we get from (1.2) and (1.4)

$$\sum m u_r = \iiint_{(\tau)} r \frac{\partial \rho}{\partial t} d\tau - \lim_{\Delta t \to 0} \sum_{(1)} \frac{\Delta M}{\Delta t} \bar{r} + \lim_{\Delta t \to 0} \sum_{(2)} \frac{\Delta M}{\Delta t} \bar{r} .$$

(1.5).....

Let a point of the body A be taken as positive, then

$$\bar{v} = \bar{v}_A + \bar{\omega} \times (\bar{r} - \bar{r}_A), \qquad (1.6).....$$

where $\bar{v}$ = velocity of an arbitrary point, $\bar{v}_A$ = velocity of the point A, $\bar{r}_A$ radius vector of point A, $\bar{\omega}$ = angular velocity of the rotation of body. The velocities of those points of the body which coincide with centers of the inertia of the particles entering through the surface are euqal

$$\bar{v}_1 = \bar{v}_A + \bar{\omega} \times (\bar{r}_1 - \bar{r}_A), \ \bar{v}_2 = \bar{v}_A + \bar{\omega} \times (\bar{r}_2 - \bar{r}_A). \qquad (1.7).....$$

v_1 and v_2 portable (carrier) velocities of the centers of inertia of the particles entering and leaving through the surface.

We know that

$$\iiint_{(\tau)} \frac{\partial \rho}{\partial t} d\tau = \lim_{\Delta t \to 0} \sum_{(1)} \frac{\Delta M}{\Delta t} - \lim_{\Delta t \to 0} \sum_{(2)} \frac{\Delta M}{\Delta t} , \qquad (1.8).....$$

Multiplying (1.5) on the left vectorially by $-2\bar{\omega}$, we get

$$\bar{J} = -2\left(\iiint_{(\tau)} \bar{v}\frac{\partial \rho}{\partial t}\,d\tau - \lim_{\Delta t\to 0}\sum_{(1)}\frac{\Delta \bar{\mu}}{\Delta t}\bar{v} + \lim_{\Delta t\to 0}\sum_{(2)}\frac{\Delta \bar{\mu}}{\Delta t}\bar{v}\right).$$

(1.9).....

Let $\bar{Q}_1^e$ be per second inward momentum of the particles through the surface of the body, and let $\bar{Q}_2^e$ be per second outward momentum of the particles through the surface, then

$$\bar{Q}_1^e = \lim_{\Delta t\to 0}\sum_{(1)}\frac{\Delta \bar{\mu}}{\Delta t}\bar{v},\quad \bar{Q}_2^e = \lim_{\Delta t\to 0}\sum_{(2)}\frac{\Delta \bar{\mu}}{\Delta t}\bar{v}. \qquad (1.10).....$$

From (1.9) and (1.10) we get

$$\bar{J} = 2\left(\bar{Q}_1^e - \iiint_{(\tau)} \bar{v}\frac{\partial \rho}{\partial t}\,d\tau - \bar{Q}_2^e\right) \qquad (1.11).....$$

We know that

$$\iiint_{(\tau)} \bar{v}\frac{\partial \rho}{\partial t}\,d\tau = \frac{d\bar{Q}}{dt} - \frac{d^*\bar{Q}}{dt},$$

where Q is principal vector of the momentum of the body, having a variable mass. Therefore, formula (1.11) may be stated as

$$J = 2\left(\bar{Q}_1^e - \frac{d\bar{Q}}{dt} + \frac{d^*\bar{Q}}{dt} - \bar{Q}_2^e\right) \qquad (1.12).....$$

2. Formula for the Principal Vector of the reaction forces.

Consider the difference $\bar{R} - \bar{J}$, where $\bar{R}$ denotes the principal vector of the reaction forces, applied on the body of a variable mass; and $\bar{J}$ is principal vector of the cariobiso forces. From equation (1.9) in Chapter II,

$$\bar{R} - \bar{J} = - \sum_{(t)} m\bar{w}_r + \bar{F} \qquad (1.13).....$$

Here the summation is taken over all the particles moving in the body with a relative acceleration $\bar{w}_r$; m is mass of such a particle The principal vector of the impulse forces is determined by (1.11) in Chapter II,

$$\bar{F} = - \sum \frac{d\mu}{dt} \Delta \bar{u}_r,$$

where the summation is carried over all the process of impulsive change of relative velocity of the particles. The value of such a pulse change in the same process is denoted by $\Delta \bar{u}_r$, and the intensity of the diffusion of mass in this process is denoted by $\frac{d\mu}{dt}$.

As seen while deriving the equation of the motion of a point of a variable mass, the principal vector of the impulsive forces is found for the processes of the pulse (impulsive) change of the velocity of particles present in the body at time t. It is

similarly found for the particles entering the body and impulsively changing their velocity in the process. Therefore, we get

$$-\bar{F}\Delta t = \sum_{(1)} \Delta m \left[\bar{u}_r(t+\Delta t) - \bar{u}_r(t)\right] + \sum_{(t)} \Delta m (\bar{u}_r^2 - \bar{u}_r^1) + O_2, \qquad (1.14).....$$

where O_2 is infinitesimal of the second order. In (1.14) the sum $\sum_{(1)}$ is taken over the particles entering into the body in time Δt, $\sum_{(t)}$ is taken over by the particles which are present in the body at time t and whose relative velocities change impulsively by $\bar{u}_r^2 - \bar{u}_r^1$ in time t.

Moreover, the vecter $\bar{\alpha}_r \Delta t$ gives continuous increment in the relative velocity of a particle. It can be determined upte the second order infinitesimal. Absolute continuous increment in the relative velocity of the same particle is equal to

$$\Delta \bar{u}_r = \bar{\alpha}_r \Delta t + (\bar{\omega} \times \bar{u}_r)\ \Delta t + O_2.$$

i.e.

$$\bar{\alpha}_r \Delta t = \Delta \bar{u}_r - (\bar{\omega} \times \bar{u}_r)\ \Delta t + O_2. \qquad (1.15).....$$

It follows that

$$\sum_{(t)} m\bar{W}_r \Delta t + \sum_{(t)} \Delta m\,(\bar{u}_r^2 - \bar{u}_r^1) =$$

$$= \sum_{(t)} m\,\Delta u_r - \sum_{(t)} m(\bar{\omega} \times \bar{u}_r)\,\Delta t + O_2. \qquad (1.16).....$$

In (1.16) the summation is taken over by all the particles found in the body at time t; Δu_r represents the total absolute increment in the relative velocity of the above particles.

$$\Delta \bar{u}_r = \bar{u}_r(t + \Delta t) - \bar{u}_r(t).$$

Using (1.13), (1.14) and (1.16) we find

$$-(\bar{R} - \bar{J})\Delta t = \sum m\left[\bar{u}_r(t + \Delta t) - \bar{u}_r(t)\right] - \sum m(\bar{\omega} \times \bar{u}_r)\Delta t +$$

$$+ \sum_{(1)} \Delta m\left[\bar{u}_r(t + \Delta t) - \bar{u}_r(t)\right] + O_2. \qquad (1.17).....$$

Consider the principal vector of the relative moment of the body of variable mass, i.e.,

$$\bar{Q}^r = \sum_{(t)} m\bar{u}_r, \qquad (1.18).....$$

where the summation is taken over by all the particles which are found in the body at the time of observation t. In order to

find the value of the principal vector of the relative moment corresponding to the time $t + \Delta t$, it suffices to take the summation in (1.18) over all the particles which are found in the body at time $t + \Delta t$. Thus it is a new set of particles over which the summation is effected, and consequently we get

$$\bar{Q}^r(t + \Delta t) = \sum_{(t+\Delta t)} m\bar{u}_r(t + \Delta t) =$$

$$= \sum_{(t)} m\bar{u}_r(t + \Delta t) + \sum_{(1)} \Delta m\, \bar{u}_r(t + \Delta t) - \sum_{(2)} \Delta m\, \bar{u}_r(t + \Delta t). \qquad (1.19).....$$

$\sum_{(t)}$ is taken over by the particles composing the body t; $\sum_{(1)}$ is taken over by the particles entering into the body in time Δt, $\sum_{(2)}$ is taken over the particles which eject from the body in time Δt.

(1.18) and (1.19) transform (1.17) into

$$-(\bar{R} - \bar{J})\,\Delta t = \bar{Q}^r(t + \Delta t) - \bar{Q}^r(t) - \bar{\omega} \times \bar{Q}^r \Delta t -$$

$$- \sum_{(1)} \Delta m\, \bar{u}_r(t) + \sum_{(2)} \Delta m\, \bar{u}_r(t + \Delta t) + Q_2. \qquad (1.20).....$$

Since

$$\overline{Q}^r(t + \Delta t) - \overline{Q}^r(t) - \overline{\omega} \times \overline{Q}^r \, \Delta t = \Delta \overline{Q}^r - \overline{\omega} \times \overline{Q}^r \, \Delta t = \Delta' \overline{Q}^r,$$

where Δ' means the increment in respect of the body as Δt tends to zero. On account of (1.20) we get

$$\overline{R} - \overline{J} = Q_1^r - \frac{d'\overline{Q}^r}{dt} - \overline{Q}_2^r. \qquad (1.21).....$$

(1.21) $\frac{d'\overline{Q}^r}{dt}$ signifies the relative derivative of the relative momentum of the body in respect of time, calculated in respect of the body;

$$\overline{Q}_1^r = \lim_{\Delta t \to 0} \sum_{(1)} \frac{\Delta u}{\Delta t} \bar{u}_r(t) \qquad (1.22).....$$

is the per second inward momentum of the particles inside the surface of the body.

$$\overline{Q}_2^r = \lim_{\Delta t \to 0} \sum_{(2)} \frac{\Delta u}{\Delta t} u_r(t + \Delta t) \qquad (1.23).....$$

is the per second flow (consumption) of the relative momentum of the particles through the surface of the body.

$\overline{Q}_1^r$ and $\overline{Q}_2^r$ are thus the surface integrals of the integrand function $\frac{d\mu}{dt} u_r$. In case of $\overline{Q}_1^r$ we consider the process by which the particles enter the surface with intensity $\frac{d\mu}{dt}$ and with relative velocity $\bar{u}_r$. Entrance velocity will be taken as $\bar{u}_r$. For

$\overline{Q}_2^r$ those processes are considered by which particles leave the surface while $\bar{u}_r$ will characterize the exit velocity immediately after leaving the surface. In case, the motion of particles in the body of variable mass is steady and does not change, then $\frac{d'\overline{Q}^r}{dt} = 0$ and consequently (1.21) can be written as

$$\bar{R} - \bar{J} = \overline{Q}_1^r - \overline{Q}_2^r.$$

On account of (1.21) we get

$$\bar{R} = \overline{Q}_1^r - \frac{d'\overline{Q}^r}{dt} - \overline{Q}_2^r + \bar{J}. \qquad (1.24).....$$

In case of established process of change in mass, formula (1.12) takes the following shape;

$$\bar{R} = \overline{Q}_1^r - \overline{Q}_2^r + \bar{J}. \qquad (1.25).....$$

2. Principal Moment of Reaction Force.

1. The formula for the principal moment of the carioliso force calculated in respect of an arbitrary point A of the body of a variable mass. We will differentiate the following expression in respect of time in two ways.

$$\bar{s} = \sum_{(t)} \left\{ m(\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \right\} \qquad (2.1).....$$

Where the summation as in (1.1) is taken over by all the particles present in the body. The range of the summation will vary at different times. In (2.1) $\bar{r}$ is the radius vector of a particle for a fixed origin; m is mass of this particle; $\bar{r}_A$ is radius vector of the point A; $\bar{\omega}$ is the angular velocity of the body.

Considering $\bar{s}$ as a sum of the geometrical points of the body, we will have

$$\bar{s} = \iiint_{\tau} \rho \, (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A)\right] d\tau , \qquad (2.2).....$$

where the integration is performed over the volume of the body. As in (1.2), the domain of integration in (2.2) may be considered fixed, assuming the density to be $\rho(x, y, z, t) = 0$.

Thus we get

$$\frac{d\bar{s}}{dt} = \iiint_{\tau} \rho \, (\bar{v} - \bar{v}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A)\right] d\tau +$$

$$+ \iiint_{\tau} \rho \, (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A)\right] d\tau +$$

$$+ \iiint_{\tau} \rho \, (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{v} - \bar{v}_A)\right] d\tau +$$

$$+ \iiint_{\tau} \rho \, (r - r_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A)\right] \frac{\partial \omega}{\partial t} \, d\tau . \qquad (2.3)....$$

Now differentiating s in another way. Consider s as the moving particles which form the body, we find.

$$\frac{d\bar{s}}{dt} = \lim_{\Delta t \to 0} \frac{\bar{s}(t + \Delta t) - \bar{s}(t)}{\Delta t} .$$

As in paragraph 1 of section 1, we get

$$\frac{d\bar{s}}{dt} = \sum_{(t)} \left\{ m (\bar{u} - \bar{v}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \right\} +$$

$$+ \sum_{(t)} \left\{ m(\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \right\} +$$

$$+ \sum_{(t)} \left\{ m (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{u} - \bar{v}_A) \right] \right\} +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{(1)} \left\{ \Delta_i^{(t)} (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \right\} -$$

$$- \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{(2)} \left\{ \Delta_i^{(t)} (\bar{r} - \bar{r}_A) \times \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \right\} . \qquad (2.4)....$$

$\sum_{(t)}$ is taken over all the particles which form the body at a given time t; $\bar{u}$ is the absolute velocity of a particle; m is the mass of this particle; $\sum_{(1)}$ is taken over all the particles entering into the body in interval $\left[t, t + \Delta t \right]$, $\sum_{(2)}$ is taken over all

particles which eject from the body in time Δt.

Comparing (2.3) and (2.4) and taking into account the usefulness of the Eulers formula.

$$\bar{v} = \bar{v}_A + \bar{\omega} \times (\bar{r} - \bar{r}_A),$$

for the velocities of points of solid body, we write

$$\sum_{(t)} \left[m\bar{u}_r \times \left[\bar{\omega} \times (r - r_A) \right] \right] + \sum_{(t)} \left[m(r - r_A) \times (\bar{\omega} \times u_r) \right] =$$

$$= - \lim_{t \to 0} \sum_{(1)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] +$$

$$+ \iiint_{\tau} \left[(\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \frac{\partial \rho}{\partial t} d\tau +$$

$$+ \lim_{\Delta t \to 0} \sum_{(2)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] . \qquad (2.5).....$$

In (2.5), the expression

$$\bar{l}^e_{2A} = \lim_{\Delta t \to 0} \sum_{(1)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \qquad (2.6).....$$

is the per second influx through the surface of the drift kinetic moment of the particles about the point A which moves in a relative

translational motion in respect of system of the axes.

Expression

$$l^{e}_{2A} = \lim_{\Delta t \to 0} \sum_{(2)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \qquad (2.7).....$$

gives the per second outflow of the drift kinetic moment about a point A in similar circumstances (conditions). I^{e}_{1A} and I^{e}_{2A} are basically the surface integrals, their domain being the surface of the body.

Opening the double vector products in the first term of the left side of (2.5) and then adding and subtracting the expression.

$$\sum_{(t)} \left\{ m\bar{u}_r \left[\bar{\omega} (\bar{r} - \bar{r}_A) \right] \right\} ,$$

We write the left side of (2.5) in the form

$$- \bar{H}_A - \bar{\omega} \times \bar{I}^{r}_{A} .$$

Here

$$\bar{H}_A = - 2 \sum_{(t)} \left\{ (\bar{r} - \bar{r}_A) \times m (\bar{\omega} \times \bar{u}_r) \right\} \qquad (2.8).....$$

denotes the principal moment of the carioliso force about the point A. Alse

$$\bar{I}^{r}_{A} = \sum_{(t)} \left[(\bar{r} - \bar{r}_A) \times m\bar{u}_r \right] \qquad (2.9).....$$

is the relative kinetic moment about the point A of the particles moving in the body with a variable mass.

Therefore, by (2.5) we obtain the following expression for the principal moment of the cariliso forces about the point A,

$$\bar{H}_A = \bar{I}^e_{1A} - \bar{\omega} \times \bar{I}_A - \iiint_{(\tau)} \left[(\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \frac{\partial \rho}{\partial t} d\tau - \bar{I}^e_{2A}. \qquad (2.10).....$$

We know that

$$\iiint_{(\tau)} \left[(\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \frac{\partial \rho}{\partial t} d\tau = \frac{d\bar{l}_A}{dt} - \frac{d^*\bar{l}_A}{dt}, \qquad (2.11).....$$

where

$$\bar{l}_A = \iiint_{(\tau)} \left[(\bar{r} - \bar{r}_A) \times (\bar{v} - \bar{v}_A) \right] \rho \, d\tau$$

is the kinetic moment of the body about the point A which is in a relative translational motion in respect of a system of axes.

Taking into consideration (2.11), we write (2.10) in the form.

$$\bar{H}_A = \bar{I}^e_{1A} - \bar{\omega} \times \bar{I}^r_A - \frac{d\bar{I}_A}{dt} + \frac{d^*\bar{I}_A}{dt} - \bar{I}^e_{2A}. \qquad (2.12).....$$

A special case arises when the point A coincides with the inertial center of the body at a given time. In such a case (2.12)

is transformed into

$$\bar{H}_c = \bar{l}^e_{1c} + \bar{l}^c_c -$$

$$- \iiint_{(\tau)} \left[(\bar{r} - \bar{r}_c) \times (\bar{v} - \bar{v}_c) \right] \frac{\partial \rho}{\partial t} d\tau - \bar{I}^e_{2c}, \qquad (2.13).....$$

which in keeping with (2.11), we get

$$\iiint_{(\tau)} \left[(\bar{r} - \bar{r}_c) \times (\bar{v} - \bar{v}_c) \right] \frac{\partial \rho}{\partial t} d\tau = \frac{d\bar{l}_c}{dt} - \frac{d^e\bar{I}_c}{dt} . \qquad (2.14)...$$

2. Principal Vector of the Reaction Forces.

Take a point A of the body of variable mass as pole and consider the difference of the principal moment of reaction and carioliso forces about the point A, i.e., $\bar{M}_A - \bar{H}_A$. By (2.2) of Chapter II,

$$\bar{M}_A - \bar{H}_A = - \sum_{(t)} (\bar{r} - \bar{r}_A) \times m \ddot{\bar{r}} + \bar{K}_A, \qquad (2.15).....$$

where the summation is taken over all the particles of the body; $\bar{K}_A$ denotes the principal moment of the impulsive forces about the point A and is equal to, according to (2.3) Chapter II,

$$\bar{K}_A = - \sum_{(t)} (\bar{r} - \bar{r}_A) \times \frac{d\bar{u}_r}{dt} \Delta\bar{u}_r .$$

The summation is thus carried over all the processes of the impulsive change of a relative velocity of the particles. $\bar{K}_A$ is introduced as a fifth order integral and the integration is

performed over the surface of the sufficiently small spheres around the geometrical points of the body and then over the volume of the body.

As before,

$$-\bar{K}_A \Delta t = \sum_{(1)} \Delta\mu \; (\bar{r} - \bar{r}_A) \times \left[\bar{u}_r(t + \Delta t) - \bar{u}_r(t)\right] + \\ + \sum_{(t)} \left[\Delta\mu \; (\bar{r} - \bar{r}_A) \times (\bar{u}_r^2 - \bar{u}_r^1) + o_2\right] \quad (2.16).....$$

the symbols used in this equation have the same meaning as in earlier discussion. $\sum_{(1)}$ is taken over the particles entering into the body in time $[t, t + \Delta t]$, $\sum_{(t)}$ is carried over the particles which are present in the body at time t and whose relative velocities undergo an impulsive change of $\bar{u}_r^2 - \bar{u}_r^1$, in time Δt.

On account of (1.15) we have

$$\sum_{(t)} \left[m \; (\bar{r} - \bar{r}_A) \times \bar{u}_r\right] \Delta t + \sum_{(t)} \left[\Delta\mu (\bar{r} - \bar{r}_A) \times (\bar{u}_r^2 - \bar{u}_r^1)\right] = \\ = \sum_{(t)} \left[m \; (\bar{r} - \bar{r}_A) \times \Delta \bar{u}_r\right] - \sum_{(t)} \left[m \; (\bar{r} - \bar{r}_A) \times (\bar{\omega} \times \bar{u}_r)\right] \Delta t + o_2.$$

(2.17).....

In this, the summation is taken over all the particles which are found in the body at time t, and

$$\Delta \bar{u}_r = \bar{u}_r(t + \Delta t) - \bar{u}_r(t)$$

denotes the total absolute increment in the relative velocity of the above mentioned particles.

From the result (2.15) - (2.17) we find

$$-(\bar{M}_A - \bar{H}_A)\Delta t = \sum_{(t)} \left\{ m\,(\bar{r} - \bar{r}_A) \times \left[\bar{u}_r(t + \Delta t) - \bar{u}_r(t)\right] \right\} -$$

$$- \sum_{(t)} \left[m\,(\bar{r} - \bar{r}_A) \times (\bar{\omega} \times \bar{u}_r) \right] \Delta t +$$

$$+ \sum_{(1)} \left\{ \Delta\mu (\bar{r} - \bar{r}_A) \times \left[\bar{u}_r(t + \Delta t) - \bar{u}_r(t)\right] \right\} + 0_2. \qquad (2.18)....$$

By an analogy to (1.9) we obtain an expression for the relative kinetic moment $\bar{l}_A^r$ at time $t + \Delta t$ expressed in formula (2.9) as follows:

$$\bar{l}_A^r(t + \Delta t) = \sum_{(t)} \left\{ m \left[\bar{r} - \bar{r}_A + \Delta (\bar{r} - \bar{r}_A)\right] \times \bar{u}_r(t + \Delta t) \right\} +$$

$$+ \sum_{(1)} \left[\Delta m \left[\bar{r} - \bar{r}_A + \Delta (\dot{\bar{r}} - \dot{\bar{r}}_A) \right] \times \bar{u}_r(t + \Delta t) \right] -$$

$$- \sum_{(2)} \left[\Delta m \left[\bar{r} - \bar{r}_A + \Delta (\dot{\bar{r}} - \dot{\bar{r}}_A) \right] \times \bar{u}_r(t + \Delta t) \right] , \qquad (2.19).....$$

where $\sum_{(t)}$ is taken over the particles which make up the body at time t; $\sum_{(1)}$ is taken over the particles which join (enter) the body in time $[t, t + \Delta t]$; $\sum_{(2)}$ is taken over the particles which eject from the body in time Δt. $\Delta(\bar{r} - \bar{r}_A)$ denotes the absolute increment in radius vector of the particle with respect to the point A in time Δt. Thus the radius vector $\bar{r} - \bar{r}_A$ has value $\bar{r} - \bar{r}_A + \Delta(\bar{r} - \bar{r}_A)$ at time $t + \Delta t$. In view of (2.9), (2.18) and (2.19) we find

$$-(\bar{M}_A - \bar{H}_A)\Delta t = \bar{I}_A^r(t + \Delta t) - \bar{I}_A^r(t) -$$

$$- \sum_{(t)} \left[m\Delta(\bar{r} - \bar{r}_A) \times \bar{u}_r(t) \right] - \sum_{(t)} m(\bar{r} - \bar{r}_A) \times (\bar{\omega} \times \bar{u}_r) \, t -$$

$$- \sum_{(1)} \left[\Delta m (\bar{r} - \bar{r}_A) \times \bar{u}_r(t) \right] +$$

$$+ \sum_{(2)} \left[\Delta m (\bar{r} - \bar{r}_A) \times \bar{u}_r(t + \Delta t) \right] + 0_2. \qquad (2.20).....$$

Also $\Delta(\bar{r} - \bar{r}_A) = \Delta'(\bar{r} - \bar{r}_A) + \bar{\omega} \times (\bar{r} - \bar{r}_A)\Delta t + 0_2$, (2.21).....

where $\Delta(\bar{r} - \bar{r}_A)$ symbolises the relative increment of radius vector $\bar{r} - \bar{r}_A$ in time Δt. Therefore, restricting its value to second order infinitesimal, we have

$$\Delta(\bar{r} - r_A) = \bar{u}_r(t)\Delta t. \qquad (2.22).....$$

Also

$$\sum_{(t)} \left[m\Delta(\bar{r} - \bar{r}_A) \times \bar{u}_r(t) \right] = \sum_{(t)} \left[m \left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \times \bar{u}_r(t) \right] \Delta t + O_2.$$

We know that (2.23).....

$$\left[\bar{\omega} \times (\bar{r} - \bar{r}_A) \right] \times \bar{u}_r + (\bar{r} - \bar{r}_A) \times (\bar{\omega} \times \bar{u}_r) = \bar{\omega} \times \left[(\bar{r} - \bar{r}_A) \times \bar{u}_r \right],$$

Therefore, (2.20) assumes the form

$$-(\bar{M}_A - \bar{H}_A)\Delta t = \Delta \bar{l}_A^r - \bar{\omega} \times \bar{l}_A^r \Delta t - \sum_{(1)} \left[\Delta m(\bar{r} - r_A) \times \bar{u}_r(t) \right] +$$

$$+ \sum_{(2)} \left[\Delta m(\bar{r} - r_A) \times u_r(t + \Delta t) \right] + O_2. \qquad (2.24).....$$

Divide (2.24) by Δt and take the limit as $\Delta t \to 0$.

$$\bar{M}_A - \bar{H}_A = \bar{I}_{1A}^r - \frac{d'\bar{I}_A^r}{dt} - \bar{I}_{2A}^r. \qquad (2.25).....$$

where $\frac{d'\bar{I}_A^r}{dt}$ stands for relative derivative with respect to time of relative kinetic moment of the set of particles which are found in the body.

$$\bar{\Gamma}^{r}_{1A} = \lim_{\Delta t \to 0} \sum_{(1)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times \bar{u}_r(t) \right] \qquad (2.26).....$$

denotes the per second inward relative kinetic moment of particles about the point A, and

$$\bar{1}^{r}_{2A} = \lim_{t \to 0} \sum_{(2)} \left[\frac{\Delta \mu}{\Delta t} (\bar{r} - \bar{r}_A) \times \bar{u}_r(t + \Delta t) \right] \qquad (2.27).....$$

denotes the per second consumption of the relative kinetic moment of particles through the surface.

Moments $\bar{\Gamma}^{r}_{1A}$ and $\bar{\Gamma}^{r}_{2A}$ define the surface integrals over the surface of the body and the function to be integrated is

$$\frac{d\mu}{dt} (\bar{r} - \bar{r}_A) \times \bar{u}_r,$$

where $\frac{d\mu}{dt}$, in the case of $\bar{\Gamma}^{r}_{1A}$, is the rate (intensity) of incoming particles in the given point of the surface and whose radius vector is $\bar{r}$. While in the case of $\bar{\Gamma}^{r}_{2A}$, $\frac{\Delta \mu}{\Delta t}$ is the corresponding rate of outgoing particles through the surface.

$\frac{d'\bar{1}^{r}_{A}}{dt}$ when the process of change of mass is set up, then $= 0$, and consequently (2.25) becomes

$$\bar{M}_A - \bar{H}_A = \bar{\Gamma}^{r}_{1A} - \bar{\Gamma}^{r}_{2A} . \qquad (2.28)....$$

A particular case arises when the point A coincides with the inertial center of the body. In this even (2.25) assumes

the form:

$$\bar{M}_C - \bar{H}_C = \bar{l}^r_{1c} - \frac{d'\bar{l}^r_c}{dt} + \bar{l}^r_{2c}. \qquad (2.29).....$$

From (2.25) we get:

$$\bar{M}_A = \bar{I}^r_{1A} - \frac{d'\bar{I}^r_A}{dt} - \bar{l}^r_{2A} + \bar{H}_A. \qquad (2.30).....$$

The principal moment of Carioliso forces is given by (2.12).

In case of established process of change in mass, formula (2.30) takes the following shape:

$$\bar{M}_A = \bar{l}^r_{1A} - \bar{l}^r_{2A} + \bar{H}_A. \qquad (2.31).....$$

3. Motion of the Center of Inertia in a body of Variable Mass.

1. Relative Velocity of the motion of inertial center.

In section $\S 1$ of Chapter II, a result was obtained by setting up a relationship between the velocities of the inertial center in the two types of systems, viz, a system with a variable mass and a solid (consolidated) system.

$$\bar{v}_c - \frac{d^*\bar{r}_c}{dt} = \frac{1}{m}\sum_j m_j(\bar{r}_j - \bar{r}_c). \qquad (3.1).....$$

In this, the summation is taken over all the material points which build up the mechanical system under observation.

Differentiating the radius vector of the inertial center in respect of time when the mass is fixed, we get:

$$\frac{d^*\bar{r}_c}{dt} = \frac{1}{m} \sum_j m_j \bar{v}_j. \qquad (3.2).....$$

Let $\bar{v}_c^e$ denote the drift (transport) velocity of the inertial center of the body. It means the velocity of the geometrical point of the body which coincides with the inertial center at a given time.

According to Euler's formula, the velocity of j-th point is given by:

$$\bar{v}_j = \bar{v}_c^e + \bar{\omega} \times (\bar{r}_j - \bar{r}_c), \qquad (3.3).....$$

where $\bar{\omega}$ is the angular velocity of the rotation of the body, r_j and r_c respectively denote the radius vectors of the point under observation and of the inertial center of the body.

In view of (3.2) and (3.3), in the case of a solid body we have:

$$\frac{d^*\bar{r}_c}{dt} = \bar{v}_c^e. \qquad (3.4).....$$

In case of a body with a variable mass, (3.1) is, therefore, transformed into:

$$\dot{v}_c = \dot{v}_c^e + \frac{1}{m} \iiint (r - r_c) \frac{\partial \rho}{\partial t} d\tau . \qquad (3.5).....$$

Here (x, y, z, t) characterizes the density of the body.

The integral is taken over the volume of body. This gives:

$$\bar{v}_c^r = \frac{1}{m} \iiint (\bar{r} - \bar{r}_c) \frac{\partial \rho}{\partial t} d\tau , \qquad (3.6).....$$

$\bar{v}_c^r = \bar{v}_c - \bar{v}_c^e$ is the called the velocity of the inertial center with respect to the frame of reference the body.

2. Relative and Carioliso Acceleration of the Motion of the Inertial Center.

Motion of the center of the inertia of a body with a variable mass with reference to the frame of the body may be characterized by the carioliso and the relative accelerations. We will, therefore, focus our attention on these and find expressions for them.

We know that the carioliso acceleration is equal to:

$$w_c^{car} = 2 \bar{\omega} \times \bar{v}_c^r . \qquad (3.7).....$$

Taking into consideration (3.3), (3.6) and (3.7) we obtain:

$$\bar{w}_c^{car} = \frac{2}{m} \iiint_{(\tau)} (\bar{v} - \bar{v}_c^e) \frac{\partial \rho}{\partial t} d\tau . \qquad (3.8).....$$

The absolute derivative of a relative velocity is equal to:

$$\frac{d\bar{v}_c^r}{dt} = \bar{w}_c^r + \bar{\omega} \times \bar{v}_c^r \,. \qquad (3.9).....$$

where $\bar{w}_c^r$ is relative acceleration of the inertial center of the body.

Differentiating (3.6) in respect of time, we get:

$$\frac{d\bar{v}_c^r}{dt} = -\frac{m}{m^2}\iiint_{\tau} (\bar{r} - \bar{r}_c)\frac{\partial \rho}{\partial t}\,d\tau + \frac{1}{m}\iiint_{\tau} (\bar{r} - \bar{r}_c)\frac{\partial^2 \rho}{\partial t^2}\,d\tau +$$

$$+ \frac{1}{m}\iiint_{\tau} (\bar{v} - \bar{v}_c)\frac{\partial \rho}{\partial t}\,d\tau \,. \qquad (3.10).....$$

In view of (3.5) and (3.6), it can be written as:

$$\frac{d\bar{v}_c^r}{dt} = \frac{1}{m}\iiint_{\tau} (\bar{r} - \bar{r}_c)\frac{\partial^2 \rho}{\partial t^2}\,d\tau - 2\frac{m}{m}\bar{v}_c^r + \bar{\omega} \times \bar{v}_c^r. \qquad (3.11)....$$

From (3.9) and (3.11) the relative acceleration of the inertial center is found to be:

$$\bar{w}_c^r = \frac{1}{m}\iiint_{\tau} (\bar{r} - \bar{r}_c)\frac{\partial^2 \rho}{\partial t^2}\,d\tau - 2\frac{m}{m}\bar{v}_c^r. \qquad (3.12).....$$

The total acceleration of the center of inertia is equal to:

$$\bar{W}_c = \bar{W}_c^e + \bar{W}_c^r + \bar{W}_c^{car} \qquad (3.13).....$$

Hence $\bar{W}_c^e$ in the drift acceleration of the inertial center or the acceleration of the geometrical point of a solid body which concides with the center of inertia at a given time.

Uniting (3.8), (3.12) and (3.13) we have W

$$\bar{W}_c = \bar{W}_c^e + \frac{1}{m} \iiint_{(\tau)} \left[(\bar{r} - \bar{r}_c) \frac{\partial^2 \rho}{\partial t^2} + 2 (\bar{v} - \bar{v}_c) \frac{\partial \rho}{\partial t} \right] d\tau .$$

(3.14).....

(3.14) can also be obtained from the relation linking the accelerations of the centers of inertia in the two systems, viz., system with a variable mass and a consolidatd (rigid) system (Section 1, Chapter II).

Actually, in view of (3.3), accelerations of the geometrical points of the body are equal to:-

$$\bar{W}_j = \bar{W}_c^e + \bar{\omega} \times (\bar{v}_j - \bar{v}_c) + \bar{\varepsilon} \times (\bar{r}_j - \bar{r}_c), \qquad (3.15).....$$

where $\bar{\varepsilon} = \frac{d\bar{\omega}}{dt}$ denotes angular acceleration.

Since the accelerations of the inertial center of a rigid system are expressed in the form:-

$$\frac{d^{2*}\bar{r}_c}{dt^2} = \frac{1}{m} \sum_j m_j \bar{w}_j,$$

Therefore, by (3.15) and (3.3) we find

$$\frac{d^{2*}\bar{r}_c}{dt^2} = \bar{w}_c^e \qquad \text{(3.16).....}$$

Applying (3.16), we can obtain (3.14) directly from (1.21) of Chapter II.

3. Mesherky's case: In Mesherky's case, the process of change of mass takes place only in surface points of the body, the integral in (3.6) must be computed over the surface of the body. Consequently, we get:

$$\bar{v}_c^r = \frac{1}{m} \iint_{(S)} (\bar{r} - \bar{r}_c) \frac{\partial \nu}{\partial t} ds. \qquad \text{(3.17).....}$$

In this, $\nu(x, y, z, t)$ is the density of the surface layer and ds is the surface lement.

Accordingly we have.

$$\bar{w}_c^{car} = \frac{2}{m} \iint_{(S)} (\bar{v} - \bar{v}_c^e) \frac{\partial \nu}{\partial t} ds, \qquad \text{(3.18).....}$$

$$\bar{w}_c^r = \frac{1}{m} \iint_{(s)} (\bar{r} - \bar{r}_c) \frac{\partial^2 \rho}{\partial t^2} \, ds - 2 \frac{\dot{m}}{m} \bar{v}_c^r. \qquad (3.19).....$$

Results (3.17), (3.18) and (3.19) enable us to estimate the velocity and the acceleration of the motion of the inertial center of the body with a variable mass, when the process of change of mass takes place in the surface points.

4. Angular velocity of the rotation of the Principal Axes of Inertia.

1. The components of velocity of the rotation of the principal axes of inertia in a body with a variable mass.

In the book on the mathematical theory of the motion of uncontrolled rockets, R.A. Rankin (refers to) (82) a result for the turning velocity of the principal inertial axes.

$$\bar{\Omega} = \iiint_{(\tau)} \left(\frac{yz\bar{I}}{I_z - I_y} + \frac{zx\bar{J}}{I_x - I_z} + \frac{my\bar{k}}{I_y - I_x} \right) \frac{\partial \rho}{\partial t} \, d\tau, \qquad (4.1).....$$

where I_x, I_y, I_z are the principal moments of the inertia and $\bar{I}$, $\bar{J}$, $\bar{k}$ are unit vectors in the directions of the principal axes x, y, z. Let us explain the method of obtaining the result (4.1).

Let us consider a system of coordinates (x, y, z) firmly connected with the body of a variable mass. Let its origin be placed at any point of the body. Then the moment of inertia of the body about the axis ℓ, passing through the origin and having direction casines α, β and γ is equal to, in consonance with the

transformation of the tensor components of inertia,

$$I_1 = I_x \alpha^2 + I_y \beta^2 + I_z \gamma^2 - 2I_{yz} \beta\gamma - 2I_{xz} \gamma\alpha - 2I_{xy} \alpha\beta \ . \qquad (4.2)....$$

The above result uses the tensor components of the inertia of the body. If the principal axis of the inertia is taken as 1, then we must have

$$\delta I_1 = 0 \qquad (4.3).....$$

for any variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, satisfying the condition

$$\delta (\alpha^2 + \beta^2 + \gamma^2) = 0. \qquad (4.4).....$$

Rewriting (4.3) and (4.4), we get

$$(I_x \alpha - I_{xy} \beta - I_{xz} \gamma) \delta\alpha + (-I_{xy} \alpha + I_y \beta - I_{yz} \gamma) \delta\beta +$$

$$+ (- I_{xz} \alpha - I_{yz} \beta + I_z \gamma) \qquad = 0, \qquad (4.5).....$$

$$\alpha \delta\alpha + \beta \delta\beta + \gamma \delta\gamma = 0. \qquad (4.6).....$$

From above, it follows that:

$$\frac{I_x \alpha - I_{xy} \beta - I_{xz} \gamma}{\alpha} = \frac{I_{xy} \alpha + I_y \beta - I_{yz} \gamma}{\beta} = \frac{- I_{xz} \alpha - I_{yz} \beta + I_z \gamma}{\gamma} \ .$$

$$(4.7).....$$

This gives

$$(I_x\alpha - I_{xy}\beta - I_{xz}\gamma)\beta = (-I_{xy}\alpha + I_y\beta - I_{yz}\gamma)\alpha \qquad (4.8).....$$

Differentiating (4.8) in respect of time, we have

$$(\dot{I}_x\alpha - \dot{I}_{xy}\beta - \dot{I}_{xz}\gamma + I_x\dot{\alpha} - I_{xy}\dot{\beta} - I_{xz}\dot{\gamma})\beta + (I_x\alpha - I_{xy}\beta - I_{xz}\gamma)\dot{\beta} =$$

$$= (-\dot{I}_{xy}\alpha + \dot{I}_y\beta - \dot{I}_{yz}\gamma - I_{xy}\dot{\alpha} + I_y\dot{\beta} - I_{yz}\dot{\gamma})\alpha +$$

$$+ (-I_{xy}\alpha + I_y\beta - I_{yz}\gamma)\dot{\alpha} \quad . \qquad (4.9).....$$

Equation (4.9) induces a dependence upon the velocity of change of the moments of inertia, calculated with reference to a fixed system of coordinates, and the velocity of the change of direction cosines of the principal axis of the inertia.

Now, we arrange the system of coordinates in such a way that at a given time the axes x, y, z, coincide with the principal axes of the inetria of the body. Therefore, at a given time the tensor of inertia and its derivative in respect of time are equal to:

$$I = \begin{Vmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{Vmatrix} \quad , \quad \dot{I} = \begin{Vmatrix} \dot{I}_x & -\dot{I}_{xy} & -\dot{I}_{xz} \\ -\dot{I}_{xy} & \dot{I}_y & -\dot{I}_z \\ -\dot{I}_{xz} & -\dot{I}_{yz} & \dot{I}_z \end{Vmatrix} \quad ,$$

where I_x, I_y, I_z are the principal moments of inertia; I_{xy}, I_{xz}, I_{yz} are centrifugal moments of inertia which at a given time are equal to zero. Moreover, when the inertia axis

coincides with x-axis we have;-

$$\alpha = 1, \beta = 0, \gamma = 0.$$

In this case, (4.9) gives.

$$\dot{\beta} = \frac{\dot{I}_{xy}}{B - A} . \qquad (4.10).....$$

Consider a geometrical point at unit distance from the origin and lying on the axis l whose direction cosines are α, β and γ. When the principal axes of the inertial rotate, this point will move in respect of the system of coordinates and its velocity is equal to

$$\bar{v} = \dot{\alpha}\,\bar{I} + \dot{\beta}\,\bar{J} + \dot{\gamma}\,\bar{k} = \bar{\Omega} \times \bar{e} = \begin{vmatrix} \bar{I} & \bar{J} & \bar{k} \\ \Omega_x & \Omega_y & \Omega_z \\ \alpha & \beta & \gamma \end{vmatrix} \qquad (4.11).....$$

where $\bar{I}$, $\bar{J}$, $\bar{k}$ are the unit vectors directed along the axes x, y, z; $\bar{e}$ is a unit vector in the direction of l; $\bar{\Omega}$ is the angular velocity of rotation of the principal axes; $\Omega_x, \Omega_y, \Omega_z$ are component of $\bar{\Omega}$ along the axes x, y, z respectively. Assuming that $\alpha = 1, \beta = 0, \gamma = 0$ (4.11) reduces to

$$\dot{\alpha}\,\bar{I} + \dot{\beta}\,\bar{J} + \dot{\gamma}\,\bar{k} = \Omega_z \bar{J} - \Omega_y \bar{k} .$$

Comparing the two sides, we find

$$\dot{\beta} = \Omega_z \cdot \qquad (4.12).....$$

Uniting (4.10) or (4.12), we have

$$\Omega_z = \frac{I_{xy}}{I_y - I_x} \cdot$$

Similarly, it will be seen that

$$\Omega_x = \frac{I_{yz}}{I_z - I_y}, \quad \Omega_y = \frac{I_{xz}}{I_x - I_z} \cdot$$

Hence, the vector giving the angular velocity of rotation of the principal axes of inertia is equal to:

$$\bar{\Omega} = \frac{I_{yz}}{I_z - I_y}\bar{I} + \frac{I_{xz}}{I_x - I_z}\bar{J} + \frac{I_{xy}}{I_y - I_x}\bar{K} \cdot$$

This leads us to the result (4.1).

In working out (4.1) we assumed that the origin of the system of coordinates coincides with some point of the body. It may happen that this point coincides with the inertial center at a given time or be itself the center of the inertia if it does not change its position in the bo

2. Mesherky's case: In the absence of the internal motion of particles, the volume integral in (4.1) must be replaced by the double integral over the surface of the body. Thus

$$\bar{D} = \iint_{(S)} \left(\frac{yz\bar{i}}{I_z - I_y} + \frac{zx\bar{j}}{I_x - I_z} + \frac{xy\bar{k}}{I_y - I_x} \right) \frac{\partial \bar{\gamma}}{\partial t} \, ds.$$

(4.13).....

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CHAPTER VI

MOTION OF THE GYROSCOPIC BODIES WITH VARIABLE MASSES.

In many problems related to modern technology it is important to investigate the notion of the fast rotating Kinetic (ally) symmetrical bodies with variable masses. We will consider a few problems relating to the motion of such bodies in a model set up.

1. Regular Precession of the Gyroscope of Variable Mass.

1. Equation of notion. Gyroscope of variable mass means a body of variable mass which has a fixed point and possess the kinetic symmetry and whose principal directions remain invariant (are conserved). The masses of the points of the body will be taken as explicit functions of time. The process of change of mass will be treated as of Mesherky's type (i.e., the particles joint or seperate (eject) from the surface). It will be assumed that the principal vector of the reaction forces is equal to zero and the principal moment of reaction forces about the fixed point of the gyroscope is directed along the axis of the symmetry and is expressed by a function of time, say, K(t).

Let θ - be the angle of nutation, ψ - angle of precession and φ - angle of self rotation of the gyroscope. Then the kinetic energy is equal to:-

$$T = \frac{1}{2}\left[A(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + C(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right],$$

where A and C are the moments of inertia of the gyroscope and they are known functions of time.

The virtual work of the usual forces is:

$$mgl \sin\theta\,\delta\theta \quad = - \delta^{*} \ (mgl \cos\theta).$$

In this δ^{*} - is a symbol of variation for the computation of which mass is supposed to be fixed. Obviously, the force function of the gravity has the form:

$$U = - mgl \cos\theta,$$

where, mgl is known function of time.

Set up virtual work of the reaction force,

$$K(t)\ (\delta\varphi + \cos\theta\,\delta\psi),$$

where K(t) - is principal moment of the reaction forces.

In the present case, there exists a special Lagranges function:

$$L_1 = T + U = \frac{1}{2}\left[A\ (\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + C\ (\dot{\varphi} + \dot{\psi}\cos\theta)^2\right] - mgl \cos\theta \qquad (1.1)$$

and the following Lagranges' equations will be useful:

$$\frac{d^*}{dt}\frac{\partial^* L_1}{\partial \dot\theta} - \frac{\partial^* L_1}{\partial \theta} = 0,$$

$$\frac{d^*}{dt}\frac{\partial^* L_1}{\partial \dot\psi} - \frac{\partial^* L_1}{\partial \psi} = K(t)\cos\theta, \qquad (1.2)$$

$$\frac{d^*}{dt}\frac{\partial^* L_1}{\partial \dot\varphi} - \frac{\partial^* L_1}{\partial \varphi} = K(t).$$

Using (1.1), we write (1.2) in detail:

$$A\ddot\theta - A\dot\psi^2 \sin\theta\cos\theta + C(\dot\varphi + \dot\psi\cos\theta)\dot\psi\sin\theta = mgl\sin\theta,$$

$$A\frac{d}{dt}(\dot\psi\sin^2\theta) + C\frac{d}{dt}\left[(\dot\varphi + \dot\psi\cos\theta)\cos\theta\right] = K(t)\cos\theta,$$

$$C\frac{d}{dt}(\dot\varphi + \dot\psi\cos\theta) = K(t). \qquad (1.3)$$

Equations (1.3) are rewritten as:

$$n(t) = \dot\varphi + \dot\psi\cos\theta = n_0 + \int_0^t \frac{K(t)}{C(t)}\,dt, \qquad (1.4)$$

$$A\ddot\theta - A\dot\psi^2\sin\theta\cos\theta + Cn\dot\psi\sin\theta = mgl\sin\theta, \qquad (1.5)$$

$$A\frac{d}{dt}(\dot\psi\sin^2\theta) - Cn\sin\theta\dot\theta = 0. \qquad (1.6)$$

2. <u>Investigation of Motion. Let us Study the Motion Confirming to the Conditions</u>:

$$\theta = \theta_0 = \text{const}, \quad \dot\psi = \dot\psi_0 = \text{const}, \qquad (1.7)$$

This type of motion is called a regular precession of the gyroscope of a variable mass. When such a motion takes place, the angular velocity of self rotation is given by:

$$\dot{\varphi}_{reg} = \dot{\varphi}^{o}_{reg} + \int_{o}^{t} \frac{K(t)}{C(t)}\, dt. \qquad (1.8)$$

If we exclude the case when $\sin \theta_o = 0$, equation (1.5) assumes the form:

$$A\dot{\psi}_o^2 \cos \theta_o - Cn\dot{\psi}_o + mgl = 0. \qquad (1.9)$$

Equation (1.6) is automatically satisfied.

Formula (1.9) determines the condition of regular precession. Suppose θ_o and $\dot{\psi}_o$ satisfy the relation (1.9) at some time and then it is also satisfied by θ_o and $\dot{\psi}_o$ in a small interval of time. Then the system of the differential equations of the gyroscope (1.3) has a unique solution (1.7) - (1.8).

We will show that the regular precession has the character of stable motion of the gyroscope of variable mass. Set up:

$$\alpha = \frac{mgl}{A}, \quad \beta = \frac{Cn}{A},$$

then (1.5) and (1.6) b come:

$$\ddot{\theta} - \dot{\psi}^2 \sin\theta \cos\theta + \beta\dot{\psi}\sin\theta = \alpha \sin\theta, \qquad (1.10)$$

$$\frac{d}{dt}(\dot{\psi}\sin^2\theta) - \beta\dot{\theta}\sin\theta = 0. \qquad (1.11)$$

Writing condition (1.9) in the new rotation, we have:

$$\dot{\psi}_0^2 \cos\theta_0 - \beta\dot{\psi}_0 + \alpha = 0. \qquad (1.12)$$

Put in (1.10) value of α from (1.12) and $\beta \sin\theta$ from (1.11), we get:

$$\dot{\theta}\ddot{\theta} - \dot{\psi}^2\dot{\theta}\sin\theta\cos\theta + (\dot{\psi}-\dot{\psi}_0)\frac{d}{dt}(\dot{\psi}\sin^2\theta) + \dot{\psi}_0^2\dot{\theta}\sin\theta\cos\theta_0 = 0. \qquad (1.13)$$

Equation (1.13) is equivalent to:

$$\frac{d}{dt}\left[\dot{\theta}^2 + \dot{\psi}(\dot{\psi}-2\dot{\psi}_0)\sin^2\theta - 2\dot{\psi}_0^2\cos\theta_0\cos\theta\right] = 0.$$

From here we get (1.7) the integral of the system (1.10) - (1.11) which vanishes in the form:

$$V = \dot{\theta}^2 + \dot{\psi}(\dot{\psi}-2\dot{\psi}_0)\sin^2\theta + \dot{\psi}_0^2\sin^2\theta_0 + 2\dot{\psi}_0^2\cos\theta_0(\cos\theta_0 - \cos\theta). \qquad (1.14)$$

Put

$$\theta = \theta_0 + \xi, \quad \psi = \psi_0 + \eta, \quad \varphi = \varphi_{reg} + \zeta. \qquad (1.15)$$

Integral (1.14) assumes the form:

$$\dot{\xi}^2 + (\dot{\psi}_0^2\xi^2 + \eta^2)\sin^2\theta_0.$$

Therefore, for considerably small values of ξ, η and $\dot{\xi}$ the integral (1.14) in a positive function and the stability corresponding to these values follows from Lyapnob's fundamental theorem.

On account of (1.15) and (1.8), equation (1.4) is written as:

$$\zeta = \zeta_0 + (\dot{\psi}_0 + \eta_0)\cos(\theta_0 + \xi_0) - (\dot{\psi}_0 + \eta)\cos(\theta_0 + \xi).$$

Hence,

$$\zeta = \zeta_0 + \left[(\dot{\psi}_0 + \eta)\sin(\theta_0 + \xi)\right]_{cp}(\xi - \xi^0) - \left[\cos(\theta_0 + \xi)\right]_{cp}(\eta - \eta_0 \quad (1.16)$$

where the index "cp" indicates the cp symbol of averaging. It is seen from (1.16) that the stability corresponding to the values and ξ ensures stability corresponding to the value ζ.

2. Investigation About the Stability of the vertical Position of the Gyroscope of a Variable Mass.

The present paragraph examines, from a purely theoretical point of view, stability condition of the vertical position of the gyroscope in which the internal motion of the particles is assumed.

1. Equations of small oscillations of the gyroscope around the vertical direction. We will obtain the equation of "apex" of gyroscope. Let us suppose that only the ejection of particles takes place (without addition).

The law of change of the kinetic moment of the body about the fixed point with possible internal motion of the particles is given by:

$$\frac{d^*}{dt}\,\bar{I} = \bar{L} + \bar{M}. \qquad (2.1)$$

where $\bar{I} = A(\omega_x \bar{i} + \omega_y \bar{j}) + C\omega_z \bar{k}$ - is the kinetic moment; $\bar{i}$, $\bar{j}$, $\bar{k}$; A and C - are moments of inertia; $\omega_x, \omega_y, \omega_z$ - are the projections of the angular velocity; $\frac{d^*}{dt}$ - stands for the derivative in respect of time for the computation of which masses are supposed to be fixed. $\bar{L}$ and $\bar{M}$ are moments of action forces and of those forces which are brought into play by a variation of mass.

Denote by $\bar{v}$ the velocity of the terminus of the vector $\bar{k}$ (i.e. "apex" of gyroscope), we got:

$$\omega_x \bar{i} + \omega_y \bar{j} = \bar{k} \times \bar{v}. \qquad (2.2)$$

Therefore, the kinetic moment is written as:

$$\bar{I} = A(\bar{k} \times \bar{v}) + C\omega_z \bar{k}. \qquad (2.3)$$

Hence equation (2.1) becomes:

$$A(\dot{\bar{k}} \times \bar{v}) + C\dot{\omega}_z \bar{k} + C\omega_z \bar{v} = \bar{L} + \bar{M}. \qquad (2.4)$$

The moments of action forces include the moments of gravity and the moments of resistance to the motion of the gyroscope including damping forces of the separated (ejected) particles. Neglecting the friction and taking into account (2.2) we have

$$\bar{L} = h\bar{k} \times m\bar{g} - \nu_1 \bar{k} \times \bar{v} - {}_2\omega_z \bar{k}. \qquad (2.5)$$

In this, h - is the distance of the center of gravity from the suspension point (Value of h will be greater than zero, equal to

zero or less than zero depending upon the center of gravity lying above the fixed point or coinciding with it or lying below the suspension point); m(t) - mass; ν_1 - is a positive function of time and ν_2 - is a positive constant (ν_1 and ν_2 characterize resistance and damping). When the rotation is rapid, the resistance $\nu_2 \omega_z$ may be replaced by an experimentally determined resistance function $f(\omega_z)$.

It is not difficult to construct such a gyroscope in which the relative kinetic moment of the particles moving in the gyroscope is zero. Taking into consideration the results obtained in Chapter V, the moment $\bar{M}$ in case of such a construction is written in the form:

$$\bar{M} = -\bar{I}^r - \bar{I}^e - \frac{d\bar{l}}{dt} + \frac{d^*\bar{I}}{dt}, \qquad (2.6)$$

Where $\bar{I}^r$ - is the per second outflow (consumption) of the rel: .ve kinetic moment of the particles through the surface of the g scope; $\bar{I}^e$ - is the per second outflow of the drift kinetic moment :he particles through the same surface.

The process by which the particles rush out is assumed to be symmetric. Therefore, we have:

$$\bar{I}^r = K\bar{k}.$$

where K - is a known function of time.

Moreover,

$$\frac{d\bar{I}}{dt} - \frac{d^*\bar{I}}{dt} = \dot{A}\,(\bar{k} \times \bar{v}) + \dot{C}\omega_z\bar{k}, \tag{2.7}$$

$$\bar{I}^e = \mu_1\,(\bar{k} \times \bar{v}) + \mu_2\omega_z\bar{k}. \tag{2.8}$$

In (2.8) μ_1 and μ_2 are known function of time which correspond to the inertial moments of per second consumption of mass. Let us examine the gyroscopes for which $\mu_1 \geqslant |\dot{A}|$, $\mu_2 \geqslant |\dot{C}|$. The functions of time K, μ_1 and μ_2, can be experimentally found like A, C, h, m and ν_1.

The multiplying equation (2.4) scalarly and vectorially by $\bar{k}$ and using (2.5) and (2.8) we get:

$$C\dot{\omega}_z + (\mu_2 + \nu_2 + \dot{C})\,\omega_z = K, \tag{2.9}$$

$$A\dot{\bar{v}} = C\omega_z(\bar{k} \times \bar{v}) - Av^2\bar{k} + h\bar{k} \times (\bar{k} \times m\bar{g}) - (\mu_1 + \nu_1 + \dot{A})\bar{v}. \tag{2.10}$$

Equation (2.10) in agreement with the terminology used by is called equation of the apex of gyroscope.

Equation (2.9) has solution:

$$\omega_z = e^{-\int_0^t \frac{\mu_2 + \nu_2 + \dot{C}}{C}\,dt} \left(\omega_{z_0} + \int_0^t \frac{K}{C}\, e^{\int_0^t \frac{\mu_2 + \nu_2 + \dot{C}}{C}\,dt}\,dt\right). \tag{2.11}$$

When the gyroscope rotates rapidly, resistance is no more linear (does not act linearly) and equation of rotation round the actual axis may be appropriately integrated.

It is easy to see that the equation (2.10) has a particular solution:

$$\bar{k} = \bar{a},\ \bar{v} = 0 \qquad (2.10)$$

where $\bar{a}$ - is unit vector of vertical axis.

Let us discuss the stability of the vertical position (2.12). In order to do this, we will obtain equations of the first approximation. Put $\bar{k} = \bar{a} + \bar{R}$. Take a fixed system of coordinates (ξ, η, ζ), which has its origin at the end of the vector $\bar{a}$. Let the axes $O\xi$ and $O\eta$ be the horizontal and let $O\zeta$ be vertical downward.

We have

$$\bar{v} = \dot{\bar{R}},\ \dot{\bar{v}} = \ddot{\bar{R}},$$

$$h\bar{k} \times (k \times m\bar{g}) = mgh\left[\bar{a}\,(R^2 + \bar{a}\bar{R}) - \bar{R}\,(1 + \bar{a}\bar{R})\right]. \qquad (2.13)$$

The project equation (2.10) on $O\xi$ and $O\eta$, on first approximation in respect of and and their derivatives, we get:

$$\ddot{\xi} = -\gamma\dot{\xi} - \beta\dot{\eta} + \alpha\xi,$$
$$\ddot{\eta} = \beta\dot{\xi} - \gamma\dot{\eta} + \alpha\eta, \qquad (2.14)$$

where $\alpha = \frac{mgh}{A}$, $\beta = \frac{C\omega_z}{A}$, $\gamma = \frac{\mu_1 + \nu_1 + \dot{A}}{A}$ are bounded functions of time.

For ζ we have:

$$= 1 - \overline{1 - \xi^2 - \eta^2}.$$

Therefore, it follows that the motion in the direction ξ , η and ζ is also stable.

D I S C U S S I O N

2. General development about the stability of motion. Multiplying the first equation of the system (2.14) by $\dot{\xi}$, and second by $\dot{\eta}$ and then adding them together we get:

$$\frac{d}{dt} V = -2\gamma (\dot{\xi}^2 + \dot{\eta}^2) - \alpha(\xi^2 + \eta^2) \qquad (2.15)$$

where

$$V = \dot{\xi}^2 + \dot{\eta}^2 - \alpha(\xi^2 + \eta^2). \qquad (2.16)$$

Suppose that the center of gravity is situated below the fixed point, then $h < 0$ and, consequently, $\alpha < 0$. Function V will be completely determined. Since $\mu_1 \geq |A|$, to $\gamma > 0$, therefore,

. Let α be a montonic increasing function. In this case, as is seen from (2.15), the derivative $\frac{d}{dt} V$ turns out to be negatively determined. For considerably small inertial values $\frac{d}{dt} V$ remains negative, if we calculate this from the non-linear equations of the motion of gyroscope, because the terms of the second and higher order do not influence the sign of the first part in (2.15).

It is, therefore, inferred, from Lyapnob's theorem, that the vertical position of gyroscope possesses a sympotic stability.

Now, suppose the center of the gravity coincides with the fixed point, then h = 0. By Lyapnob's theorem, we will prove that a symptotic stability corresponding to the values ξ and η will also be small and for their evaluation, we may use equations of the first approximation (1.14) which yield:

$$\dot{\xi}^2 + \dot{\eta}^2 = (\dot{\xi}_0^2 + \dot{\eta}_0^2)\, e^{-2\int_0^t \gamma dt} \qquad (2.17)$$

If the process of change of mass takes place for a finite interval of time only, and if the air resistance can be neglected, then for $t \geqslant T$ the coefficients of the system (2.14) will be constant.

Characteristic Equation is Written in the Form:

$$\lambda^4 + 2\gamma\lambda^3 + (\gamma^2 - 2\alpha + \beta^2)\lambda^2 - 2\alpha\gamma\lambda + \alpha^2 = 0.$$

In order to satisfy condition, it is necessary and sufficient that $\alpha < 0$. Therefore, if h is negative at time t, the vertical position of the gyroscope will be stable. The character of the gyroscope motion may be studied by the method of B.V. Bulgakov (6).

Suppose the coefficients in equations of the system (2.14) approximate to the finite limits as $t \to \infty$. Assuming $\nu_2 = 0$, we obtain a limiting system with constant coefficients. If the limiting value α is less than zero, then the vertical position of

the gyroscope possesses stability according to the theorem of K.P. Percidsko [79].

3. Stability of Vertical Position of the Gyroscope when there is no Internal Motion of the Particles.

In the absence of internal motion of the particles in a gyroscope, for example, in case of surface heating, we have

$$\mu_1 = -\dot{A}, \quad \mu_2 = -\dot{C}.$$

Neglecting the air resistance, then in view of (2.11) and (2.14) we get:

$$\omega_z = \omega_z^0 + \int_0^t \frac{K}{C}\, dt, \tag{2.18}$$

$$\begin{aligned} \dot{\xi} &= -\beta\dot{\eta} + \alpha\xi, \\ \dot{\eta} &= \beta\dot{\xi} + \alpha\eta. \end{aligned} \tag{2.19}$$

We will first prove a theorem about the approximation of the solution of a system of the linear differential equations with variable coefficients.

Suppose we are given a linear system:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j \quad (i = 1, 2, \ldots, n), \tag{2.20}$$

where a_{ij} - are continuous functions of time and $|a_{ij}| < a$ for $0 \leqslant t < \infty$. Take an arbitrary interval of time $(0,T)$ and divide it by points $t_0, t_1, t_2, \ldots, t_{m-1} = T - \chi, t_m = T + \tau - \chi$ into m parts such that $t_{i+1} - t_i = \tau, \chi \leqslant \tau$, write the approximating system:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij} y_j \qquad (2.21)$$

where:

$$a'_{ij} = \begin{cases} a_{ij}(0) & \text{for } 0 \leqslant t < t_1, \\ a_{ij}(t_1) & t_1 \leqslant t < t_2, \\ \vdots & \vdots \\ a_{ij}(t_{m-1}) & t_{m-1} \leqslant t < T. \end{cases} \qquad (2.22)$$

Theorem then states that for arbitrarily small $\varepsilon > 0$ and $T > 0$ arbitrarily large $\tau > 0$, it is possible to choose a fairly small $\tau > 0$, such that $x_i^0 = y_i^0$ u$0 \leqslant t \leqslant T$ and

$$|x_i - y_i| < \varepsilon. \qquad (2.23)$$

The continuous solution of the system (2.21) with initial condition (Value) y_i^0 is unique. In order to find y_i apply the method of successive approximations taking x_i as the zero approximation. From (2.21) we have:

$$y_i = y_i^o + \int_o^t \sum_{j=1}^n a'_{ij} y_j dt,$$

The first approximation is written as:

$$y_{i1} = x_i + \int_o^t \sum_{j=1}^n (a'_{ij} - a_{ij})\, x_j dt.$$

It follows that:

$$\left| y_{i1} - x_i \right| < \delta A\,(T)\, nt,$$

in which we have used the property of boundedness of the integrand function:

$$\left| a'_{ij} - a_{ij} \right| < \delta, \quad \left| x_j \right| < A\,(T).$$

Further we have:

$$y_{i2} - y_{i1} = \int_o^t \sum_{j=1}^n a'_{ij}\,(y_{j1} - x_j)\, dt,$$

$$\left| y_{i2} - y_{i1} \right| < \delta \frac{A\,(T)}{a} \frac{(ant)^2}{2!}$$

Accordingly, we get:

$$\left| y_{ik} - y_{i,\,k-1} \right| < \delta \frac{A\,(T)}{a} \frac{(ant)^k}{k!}$$

As a result:

$$y_i = x_i + \sum_{n=1}^{\infty} (y_{in} - y_{i,\,n-1}),$$

$$\left| y_i - x_i \right| < \delta \frac{A\,(T)}{a} (e^{anT} - 1).$$

Since the functions a_{ij} - are continuous, therefore, in the closed interval $[0, T]$ they are uniformly continuous. Consequently, it is required to find such that:

$$\delta < \varepsilon \frac{a}{A(T)} e^{-\alpha T}.$$

This proves (2.23). It may be remarked that the above valuation will be uniform in respect of T when τ varies.

Let us discuss the stability of zero solution of the system (2.19). This is a linear system andthe theorem now now proved applies to it.

Consider the approximating system:

$$\begin{aligned} \ddot{x}_1 &= -\beta' \dot{y}_1 + \alpha' x_1, \\ \ddot{y}_1 &= -\beta' \dot{x}_1 + \alpha' y_1. \end{aligned} \tag{2.24}$$

where α' and β' - are constants obtained by the rule of approximation (2.22), Assuming $u = x_1 + iy_1$, $i = \sqrt{-1}$, the system (2.24) reduces to a single equation:

$$\ddot{u} - i\beta' \dot{u} - \alpha' u = 0. \tag{2.25}$$

For j-th interval, we have:

$$\ddot{u}_j - i\beta_j \dot{u}_j - \alpha_j u_j = 0. \tag{2.26}$$

Set up

$$u_j = w_j e^{i \frac{\beta_j}{2} t}, \tag{2.27}$$

then for the variable w_j, we find the equation:

$$\ddot{w}_j + \left(\frac{\beta_j^2}{4} - \alpha_j\right) w_j = 0. \tag{2.28}$$

(2.27) and (2.28) show that the motion of the point with coordinates x_1 and y_1 may be looked upon as combination of the relative motion in conformity with the equation (2.28) and the drift motion of the uniform rotation whose angular velocity is $\frac{\beta_j}{2}$.

We see that the stability of zero solution of the system (2.24) follows from the stability of sequence of functions w_j and $\dot{w}_j$ because, on account of (2.27) we have:

$$|u_j| = |w_j|, \quad |\dot{u}_j| = \left|\dot{w}_j + i \frac{\beta_j}{2} w_j\right|. \tag{2.29}$$

The solution in u and $\dot{u}$ must be continuous. Therefore, by virtue of (2.29) we obtain the following conditions in going from j-th interval to j + 1 -th.

$$\rho_j(\tau) = \rho_{j+1}(0), \quad \dot{\rho}_j(\tau) = \dot{\rho}_{j+1}(0), \tag{2.30}$$

where $$\omega_j(\tau) - \frac{B_j}{2} = \omega_{j+1}(0),$$

where $\rho_j = \tau\omega_j$; ω_j - is the angular velocity in the relative motion or derivative in respect of time of the argument of complex number $\tau\omega_j$; $B_j = \beta_{j+1} - \beta_j$.

Suppose $\frac{\beta_2}{4} - \alpha > 0$. Denoting this value for j-th interval by b_j^2, we obtain the obvious integrals of equation (2.28).

$$v_{rj}^2 + b_j^2 \rho_j^2 = H_j, \; H_j = \text{const}, \tag{2.31}$$

$$\omega_j \rho_j^2 = \sigma_j, \; \sigma_j = \text{const}, \tag{2.32}$$

where $v_{rj} = \sqrt{\rho_j^2 + \omega_j^2 \rho_j^2}$ - is relative velocity in j-th interval.

Equation (2.30) and (2.32) yield

$$\sigma_j = \sigma_o - \frac{1}{2} \sum_{i=o}^{j-1} B_i \rho_i^2(\tau). \tag{2.33}$$

Suppose b^2 diminishes (not strictly). Then from (2.30) - (2.33), we have

$$H_{j+1} = H_o + \sum_{i=o}^{j-1} \left(b_{i+1}^2 - b_i^2 + \frac{B_i}{2} \sum_{k=i+1}^{j} B_k\right) \rho_i^2(\tau) +$$

$$+ (b_{j+1}^2 - b_j^2) \rho_j^2(\tau) - \sigma_o (\beta_{j+1} - \beta_o) + \frac{1}{4} \sum_{i=0}^{j} B_i^2 \rho_i^2(\tau). \tag{2.34}$$

The terms in the first bracket are equal to:

$$\left(\frac{\beta_{j+1}}{2}\dot{\beta}_i - \dot{\alpha}_i\right)\tau + \left(\frac{d^2b^2}{dt^2}\Big|_i^{cp} + \ddot{\beta}\Big|_i^{cp}\right)\frac{\beta_{j+1} - \beta_{i+1}}{2}$$

$$- \frac{\beta_i}{2}\dot{\beta}\Big|_i^{cp}\Big)\,\tau^2. \qquad (2.35)$$

,,cp" denotes the average of i-th interval; $\dot{\alpha}_i$ and $\dot{\beta}_i$ - are derivatives computed at time t_{i-1}.

Since b^2 diminishes, then for sufficient average velocities we may take $\frac{\beta_{min}}{2}\dot{\beta} \ldots \alpha \leqslant 0$. Particularly for any i and j,

$$-\frac{\beta_{j+1}}{2}\dot{\beta}_i - \dot{\alpha}_i \leqslant 0. \qquad (2.36)$$

From (2.34) and (2.35), we get

$$H_{j+1} \leqslant H_o + 2\left|\sigma_o\right|\beta_{max} + \check{M}H(T)\,\tau(T + \tau),$$

where

$$\check{M} = \sup\left(\left|\frac{d^2b^2}{dt^2}\right| + \left|\ddot{\beta}\right|\left|\beta\right| + \frac{3}{4}\left|\dot{\beta}\right|^2\right),\quad H(T) = \sup \rho^2.$$

Suppose the initial values are taken to be adequately small, then, assuming that

$$\left\{v_r^o,\quad \rho_o\right\} < \delta,$$

$$\tau < \left\{\frac{\delta 2}{\check{M}H(T)\,(T + \frac{1}{2})},\ \frac{1}{2}\right\},$$

we get

$$v_r^2 + b^2 \rho^2 < (2 + b_o^2 + 2 \left|\sigma_o\right| \sup \beta)\, \delta^2. \qquad (2.37)$$

Since (2.37) does not depend upon T, therefore, it will remain true for the interval $0 \leqslant t < \infty$.

It is interesting to consider the case when β and b^2 increase (not strictly). The expression

$$\frac{v_{rj}^2}{b_j^{2p+2}} + \frac{\rho_j^2}{b_j^{2p}} = G_j, \quad p = \text{const}, \quad G_j = \text{const} \qquad (2.38)$$

will be similarly on the integral of equation (2.28) on j-th interval. By virtue of (2.30) - (2.33) and (2.38), we have

$$G_j \leqslant G_o - \sigma_o \sum_{i=o}^{j} \frac{B_i}{b_{i+1}^{2p+2}} + \frac{1}{4} \sum_{i=o}^{j} B_i^2 \frac{\rho_i^2(\tau)}{b_{i+1}^{2p+2}} + \sum_{i=o}^{j-1} S_{ij}\, \rho_i^2(\tau),$$

where

$$S_{ij} = \frac{1}{b_{i+1}^{2p}} + \frac{1}{b_i^{2p}} + \frac{B_i}{2} \left(\frac{B_{i+1}}{b_{i+2}^{2p+2}} + \frac{B_{i+2}}{b_{i+3}^{2p+2}} + \ldots + \frac{B_j}{b_{j+1}^{2p+2}} \right).$$

Since b^2 - is an increasing function, we have

$$S_{ij} \leqslant \frac{1}{b_{i+1}^{2p}} - \frac{1}{b_i^{2p}} + \frac{B_i}{2b_i^{2p+2}} \left(\beta_{j+1} - \beta_{i+1} \right).$$

and also

$$S_{ij} \leqslant \frac{1}{b_i^{2p+2}} \left[\frac{-(p+1)\beta_o + \sup \beta}{2} \beta_i + p \alpha_i \right] \tau +$$

$$+\left[\frac{d^2}{dt^2}\ \frac{1}{b^{2p}}\right|_i^{cp}+\frac{\ddot{\beta}\Big|_i^{cp}}{2b_i^{2p+2}}\left(\beta_{j+1}-\beta_{i+1}\right)\Bigg]\tau^2.$$

Lets up $\beta < q\,\beta_o$ for some number q. When the relative velocities of the separating particles are sufficiently large, we have $\dot{\beta} \gg |\dot{\alpha}|$. Therefore, we can find a sufficiently large p such that

$$-\frac{p-q+1}{2}\,\beta_o\,\dot{\beta}+p\,\dot{\alpha}\leqslant 0.$$

Hence for a sufficiently small and whenever, the inequality

$$\left\{v_r^o,\ \rho_o\right\}<\delta$$

is satisfied

$$\frac{v_r^2}{b^{2p+2}}+\frac{\rho^2}{b^{2p}}\leqslant\left(1+\frac{1+2\sup\beta}{b_o^{2p+2}}+\frac{1}{b_o^{2p}}\right)\delta^2.$$

We have shown that in the above two situation the vertical position of the gyroscope possesses stability of the first approximation.

3. Transition to Equation of the Applied Theory of the Gyroscopes for Gyroscopic System with Variable Masses.

In order to investigate the motion of a system with the gyroscopes we usually introduce the second derivatives Lagrangain coordinates in the linearized equations [7, 26-28, 38, 40].

Let us study the gyroscopic systems with variable coefficients.

The proposed method of the investigation based on the change of independent variable is quite interesting. This method envisages to use equations developed with the help of applied theory of the gyroscopes under conditions of change of mass and considerable changes in the position of the coordinates and the velocities.

1. Equations of the motion of Gyroscopic Systems with Variable Masses.

We will study the general case of the motion of gyroscopic systems, assuming the arbitrary dependence of the motion and the mass of gyroscopic system on time and also unsteady inherent rotation of the gyroscopes.

The kinetic energy of gyroscopic system with r gyroscopes and S position coordinates is

$$T = T'_2 + T'_1 + T'_0 + \frac{1}{2} \sum_{\nu=1}^{r} C_\nu \left(\varphi_\nu + \sum_{k=1}^{S} a_k^\nu q_k + a_0^\nu \right)^2. \tag{3.1}$$

where T'_2, T'_1 and T'_0 - respectively denote the quadratic linear and zero forms of the positional velocities q_i, φ_ν - are cyclic coordinates, representing the angles of the inherent rotation of the gyroscopes; C_ν - is the axial moment of the inertia of ν-th gyroscope; a_k^ν - is cosine of the angle between the vector of the angular velocity q_k and the axis of ν-the gyroscope; a_0^ν - is projection of the angular velocity of on the axis of ν-th gyroscope.

Coefficients a_k and a_o depend upon position vectors and time.

Suppose the generalized usual and reaction forces responding to the cyclic coordinates are explicit functions of time. Forming the Lagrangiam equations of the second type for the cyclic coordinates φ_ν, we have

$$\dot{\varphi}_\nu + \sum_{k=1}^{s} a_k^\nu q_k + a_o^\nu = H + h_\nu , \qquad (3.2)$$

where H - is a constant which is sufficiently large $H \gg h_\nu$, h_ν - are functions of time containing r -- 1 constants.

Introduce Roryc's function

$$\check{R}(\dot{q}_i, q_i, t) = \check{T} - \sum_{\nu=1}^{r} C_\nu (H + h_\nu)\dot{\varphi}_\nu , \quad \check{T} = T(\dot{q}_i, q_i, \dot{\varphi}_\nu(\dot{q}_i, q_i, H, T)),$$

Then Lagrangiam equations of the second type for the position coordinates can be written as:

$$\frac{d^*}{dt}\frac{\partial^* \check{R}}{\partial \dot{q}_i} - \frac{\partial^* \check{R}}{\partial q_i} = Q_i + \Psi_i . \qquad (3.3)$$

Here the derivatives are calculated by keeping the mass fixed. Ψ_i - represent the generalized reaction forces. Rayc's function in accordance with (3.1) and (3.2) is expressed

$$\check{R} = T' + \sum_{\nu=1}^{r} C_\nu (H + h_\nu)\left(\sum_{k=1}^{s} a_k^\nu q_k + a_o^\nu\right) - \frac{1}{2}\sum_{\nu=1}^{r} C_\nu (H + h_\nu)^2 , \qquad (3.4)$$

$$T' = T'_2 + T'_1 + T'_o ,$$

where the last member (term) is explicit function of time.

In (3.4), T' represents the kinetic energy of the absolute motion of elements of the gyroscopic system, casing and motors of gyrosopes, as well as kinetic energy of motion of the rotors themselves around their own axis. Let us record equation (3.3) with the help of (3.4) as follows:

$$\sum_{k=1}^{s} a_{ik}\ddot{q}_k + H\left(\sum_{k=1}^{s} g_{ik}\dot{q}_k + g_{i0}\right) + \Omega_i = 0 \qquad (3.5)$$

Here the following symbols are introduced:

$$g_{ik} = \sum_{\nu=1}^{r} C_\nu \left(\frac{\partial a_i^\nu}{\partial q_k} - \frac{\partial a_k^\nu}{\partial q_i}\right), \quad g_{i0} = \sum_{\nu=1}^{r} C_\nu \left(\frac{\partial a_i^\nu}{\partial t} - \frac{\partial a_0^\nu}{\partial q_i}\right),$$

$$\Omega_i = \check{\Phi}_i - \check{\vartheta}_i, \quad \check{\Phi}_i = \frac{d^*}{dt}\frac{\partial^* \check{T}}{\partial \dot{q}_i} - \frac{\partial^* \check{T}}{\partial q_i} - \sum_{k=1}^{s} a_{ik}\ddot{q}_k, \quad a_{ik} = \frac{\partial^2 \check{T}}{\partial \dot{q}_i \partial \dot{q}_k},$$

$$\check{\vartheta}_i = Q_i + \Psi_i - \sum_{\nu=1}^{r} C_\nu h_\nu \left[\left(\frac{\partial a_i}{\partial q_k} - \frac{\partial a_k^\nu}{\partial q_i}\right)\dot{q}_k + \left(\frac{\partial a_i^\nu}{\partial t} - \frac{\partial a_0^\nu}{\partial q_i}\right)\right], \qquad (3.6)$$

If the equations of the motion are written according to the method of applied theory of the gyroscopes, we assume $\check{T} = 0$. Denoting the position coordinates by g_i, we get

$$H\left(\sum_{k=1}^{s} g_{ik}\dot{g}_k + g_{io}\right) - \check{\Theta}_i = 0, \qquad (3.7)$$

where in g_{ik}, g_{io} the position coordinates and in $\check{\Theta}_i$ the position coordinates and the velocities are replaced by g_i and $\dot{g}_i$.

If the gyroscopic system is laid on a fixed foundation then $\partial a_j^{\nu}/\partial t = 0$, $a_o^{\nu} = 0$ and consequently $g_{io} = 0$.

In case of real gyroscopic systems on a mobile foundation, coefficient g_{io} may have an angular velocity, of fairly small magnitude in the earth's rotation. The pendulum moment, however, which is included in g_i, is close to the value H in some gyroscopic systems. Therefore, we will focus our attention on special features in each concrete system. Coefficients a_{ik}, g_{ik} and g_{io} or order H not higher than zero and all functions (3.6) factoring in power series.

We will assume that the determinant of the matrix of the gyroscopic terms $\|g_{ik}\|$ is different from zero.

Let us consider the differential equations involving the small parameter λ:

$$\sum_{k=1}^{s} g_{ik}^{(1)} \dot{q}_{k1} + g_{io}^{(1)} + \lambda \check{\Phi}_i^{(1)} - H^{-1} \check{\Theta}_i^{(1)} = 0.$$

Symbol (1) indicates that in these functions the position coordinates and velocities are replaced by variables q_{i1} and $\dot{q}_{i1}$.

In accordance with Poincrr's method of small parameter [14], the variables q_{i1} and $\dot{q}_{i1}$ are determined by equations (3.7) upto terms of order. In particular, if $\lambda = H^{-1}$, we have

$$\left\{ q_{i1} - g_i,\ \dot{q}_{ia} - \dot{g}_i \right\} = O(H^{-1}). \qquad (3.8)$$

2. Motion of the Gyroscopic System Placed on a Fixed Foundation.

In the first instance, consider a stable system and obtain the linearized equations.

When the position coordinates $q_1, \ldots, q_s$ and their derivatives are slightly disturbed, the natural oscillations of the stable gyroscopic systems are determined from the equations

$$a_{ik}\ddot{q}_k + (b_{ik} + Hg_{ik})\dot{q}_k + c_{ik}q_i = 0, \qquad (3.9)$$

in which a considerably large positive constant H represents the least value of the angular velocities of the system in respect of the gyroscopic axis. $\|a_{ik}\|$ is the matrix of positive definite quadratic form. The coefficients in the equation (3.9) are sounded functions of time and have zero order in respect of H. The stable

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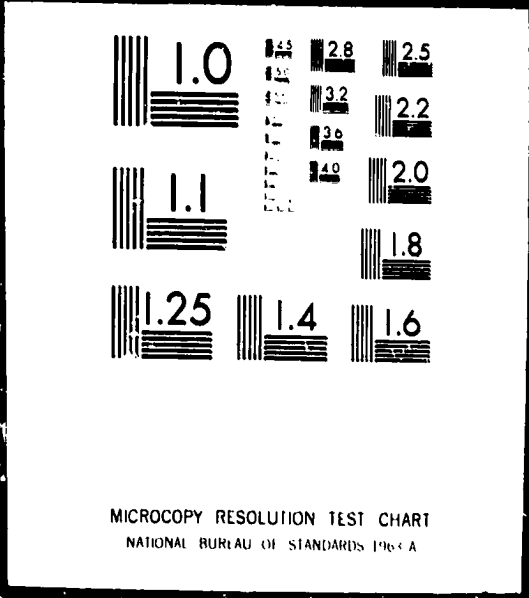
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system cannot have solutions of order higher than zero in respect of H. The indices i and k range from 1 to S, the same two indices of (in) multipliers means the summation.

We will study the solution of the two systems of equations

$$(b_{ik} + Hg_{ik})\,\dot{q}_{i1} + c_{ik}q_{k1} = 0, \tag{3.10}$$

$$a_{ik}\ddot{q}_{k2} + (b_{ik} + Hg_{ik})\,\dot{q}_{k2} = 0. \tag{3.11}$$

Since the determinant of the matrix of the gyroscopic elements is different from zero, therefore, when H is large the system (3.10) can be solved for q_{k1}. The system (3.11) can always be solved for q_{k2}.

We will show that the solution of the system (3.9) satisfying the conditions $t = 0$, $q_i = q_i^\circ$, $\dot{q}_i = \dot{q}_i^\circ$, in any finite interval of time (precisely upto the value of order H^{-1} is written in q_{i1} and q_{i2}.

Assuming the following initial conditions

$$q_{i1}^\circ = q_i^\circ, \quad \left[b_{ik}(0) + Hg_{ik}(0)\right]\dot{q}_{k1}^\circ + c_{ik}(0)\,q^\circ{}_k = 0, \tag{3.12}$$

$$q_{i2}^\circ = 0, \quad \dot{q}_{i2}^\circ = \dot{q}_i^\circ - \dot{q}_{i1}^\circ. \tag{3.13}$$

Changing to new independent variables $\tau_1 = H^{-1}t$ in (3.10) and $\tau_2 = Ht$ in (3.11) and denoting the derivatives in respect of new independent variables by "dashes" we get:

$$(H^{-1}b_{ik} + g_{ik})\, q'_{k1} + c_{ik}q_{k1} = 0,$$

$$(H^{-1}b_{ik} + g_{ik})\, q''_{k1} + (\dot{b}_{ik} + H\dot{g}_{ik} + c_{ik})\, q'_{k1} + H\dot{c}_{ik}q_{k1} = 0, \qquad (3.14)$$

$$a_{ik}q''_{k2} + (H^{-1}b_{ik} + g_{ik})\, q'_{k2} = 0,\ q_{i2} = (f_{ik}q'_{k2})_0^t - \int_0^t f_{ik}q'_{k2}dt, \qquad (3.15)$$

where f_{ik} are bounded functions of zero order in H obtained by solving the first group of equation (3.15) in respect of q'_{i2}. The derivatives f_{ik} cannot have order higher than zero in H.

The motions described by $q_i^{(1)}$ and $q_i^{(2)}$ satisfying the initial conditions (3.12) and (3.13) are called the precession and the mutation of the gyroscopic system (3.9). A solution of the system (3.9) which satisfies the general conditions is $q_i^{(1)} + q_i^{(2)}$. $q_i^{(1)} = q_{i1} + x_i$, $q_i^{(2)} = q_{i2} + y_i$. The variables x_i and y_i are obtained as the particular solutions satisfying the zero initial conditions of the equations

$$a_{ik}\ddot{x}_k + (b_{ik} + Hg_{ik})\, \dot{x}_k + c_{ik}x_k = -H^{-2}a_{ik}q''_{k1}, \qquad (3.16)$$

$$a_{ik}\ddot{y}_k + (b_{ik} + Hg_{ik})\, \dot{y}_k + c_{ik}y_k = -c_{ik}q_{k2}. \qquad (3.17)$$

It is seen from the system of equations (3.14) and (3.15) that in a finite interval of time $[0,T]$ the variables q''_{k1} have order H and q_{ik} have order H^{-1}. In accordance with (3.16) and (3.17) we get

$$\{x_{i1}, \dot{x}_i, y_i, \dot{y}_i\} = O(H^{-1}). \qquad (3.18)$$

The variables q_{i1} and q_{i2} are of the order H^{-1} and, therefore, it follows from (3.18) that

$$q_i = q_{i1} + O(H^{-1}), \quad \dot{q}_i = \dot{q}_{i2} + O(H^{-1}). \qquad (3.19)$$

We find that the values q_{i1} change slowly, while $\dot{q}_{i2}$ change rapidly for $\dot{q}_{i1} = O(H^{-1})$ and $\ddot{q}_{i2} = O(H)$.

If the system (3.9) remains stable under the steady active perturbance and the variables q_{i1} and q_{i2} are bounded, then it is seen from the systems (3.14), (3.16) and (3.17) that the results (3.18) and (3.19) also hold good in an infinite interval of time.

The forced oscillations in a stable gyroscopic system produced under the action of the generalized forces $f_i(t)$, may be substituted by a particular solution satisfying the zero initial conditions of the system of equations.

$$(b_{ik} + Hc_{ik})\dot{q}_{k3} + c_{ik}q_{k3} = f_i(t).$$

The error in coordinates as well as in the velocities is determined by a solution satisfying the zero initial conditions of the equations (3.16). In doing this q''_{k1} must be replaced by values obtained for the solution with corresponding initial conditions of the second group of the equations of the system (3.14) whose right hand side is Hf_i. Therefore, the error will have the order of q''_{k1} divided by H^2. If $f_k(t)$ possess the zero order of H, then the order of the error does not exceed H^{-1} in any interval of time in

which f_k (t) are bounded.

We will try to obtain the refinements on the valuations of (3.18) and (3.19). By introducing the independent variables $\tau_2 = Ht$ in equations (3.17) we get

$$a_{ik} y_k'' + (H^{-1} b_{ik} + g_{ik}) y_k' = - H^{-2} c_{ik} (q_{k2} + y_k). \qquad (3.20)$$

The values $q_{k2} + y_k$ have the order of H^{-1}. In order to find a particular solution of equations (3.20) with the zero initial conditions it is necessary to integrate in respect of τ_2, for the range of the integration being from zero to Ht, therefore, the indicated particular solution y_i' will have the order of H^{-2}. From equations (3.20) we find.

$$y_i = (f_{ik} y_k')_o^t - \int_o^t f_{ik} y_k' dt + 0 (H^{-2}), \; y_i = 0 (H^{-2}).$$

We will think that for a gyroscopic system the conditions (a) are satisfied, if g_{ik} and c_{ik} are equal to zero or their order does not exceed H^{-1}. Then the system (3.14) shows that q_{i1}'' possess the zero order of H. Therefore, in accordance with the equations (3.16) we get $x_i = 0 (H^{-2})$.

Thus we have shown that

$$q_i = q_{i1} + q_{i2} + 0 (H^{-2}).$$

is valid in case of a solution of equations (3.9) with arbitrary initial conditions and which satisfies the conditions (a).

Let us examine the considerable defection of a gyroscopic system placed on a fixed foundation. Suppose the generalized forces of an absolute motion contain only the terms of zero order of H.

We anticipate a solution of the form of the equations (3.5).

$$q_i = q_{i1}(H^{-1}t) + x_i(Ht). \qquad (3.21)$$

The variables q_{i1} in this case satisfy the equations

$$H\, g_{ik}^{(1)} \dot{q}_{k1} + \Omega_i^{(1)} = 0. \qquad (3.22)$$

Assume the following initial conditions

$$q_{i1}^0 = q_i^0,\ x_i^0 = 0,\ g_{ik}^{(1)\,0} q_{k1}^0 + \Omega_i^{(1)\,0} = 0,\ \dot{x}_i^0 = \dot{q}^0 - q_{i1}^0. \qquad (3.23)$$

Let us denote the derivatives for the arguments: $\tau_1 = H^{-1}t$ and $\tau_2 = Ht$ by "dashes". If $\partial g_{ik}^{(1)}/\partial t$ and $\partial \Omega_i/\partial t$ are equal to zero or their order does not exceed H^{-1}, then the gyroscopic system satisfies the conditions (a). It is seen from the equations (3.22) and the conditions (3.23) that q_{i1} and q'_{i1} possess the zero order of H. Further, q''_{i1} have an order H, but when the conditions (a) are satisfied it has zero order of H.

With the help of (3.5), (3.21), (3.22) and the equations $f_{io} = 0$ we obtain the following equations to find x_i

$$a_{ik}(q_{\ 1}(H^{-2}\tau_2) + \chi_n H^{-1}\tau_2)\, x''_k + g_{ik}(q_{\gamma_1}(H^{-2}\tau_2) + x_\gamma, H^{-1}\tau_2)\, x'_k +$$

$$+ H^{-2}\left[\Omega_i\left(q_{\gamma 1}(H^{-2}\tau_2) + x_\gamma, H^{-1}q'_{\gamma 1}(H^{-2}\tau_2) + Hx'_\gamma, H^{-1}\tau_2\right) -\right.$$

$$-\Omega_i\left(q_{\gamma 1}(H^{-2}\tau_2), H^{-1}q'_{\gamma 1}(H^{-2}\tau_2), H^{-1}\tau_2\right) +$$

$$\left. + H^{-2}\varphi_i(H^{-1}, x_\gamma, \tau_2)\right] = 0,$$

where

$$\varphi_i = a_{ik}\left[q_{\gamma 1}(H^{-2}\tau_2) - x_\gamma, H^{-1}\tau_2\right] q''_{k1}(H^{-2}\tau_2) +$$

$$+\left[G_{ik}\left(q_{\gamma 1}(H^{-2}\tau_2) + x_\gamma, H^{-1}\tau_2\right) -\right.$$

$$\left. - G_{ik}\left(q_{\gamma 1}(H^{-2}\tau_2), H^{-1}\tau_2\right)\right] q'_{k1}(H^{-2}\tau_2).$$

When the values ξ_k possess the zero order, we have

$$0\left\{\Omega_i(\xi_k, \eta_k, t) = 0(\eta_k^\sigma).\right.$$

The cases when $\sigma = 0, 1, 2,$ are of practical interest. The functions φ_i have an order H and when the conditions (a) are satisfied they have zero order in respect of H.

According to Poincre's method of small parameter [14] we get

$$\left\{x_i - y_i, x'_i - y'_i\right\} = 0(H^{-1}). \qquad (3.24)$$

The variables y_i are such that require x_i to satisfy the equations

$$a_{ik}(q^0_\gamma + y_\gamma, 0)\, y''_k + G_{ik}(q^0_\gamma + y_\gamma, 0)\, y'_k =$$

$$= -H^{-2}\left[\Omega_i(q^0_\gamma + y_\gamma, Hy'_\gamma, 0) - \Omega_i(q^0_\gamma, 0, 0)\right]. \qquad (3.25)$$

If $\sigma = \{0, 1\}$, right hand side of (3.25) vanishes. Because a_{ik}, β_{ik} and at $\sigma = 2$ the right hand side of equation (3.25) is zero to the order of μ, therefore, in the line with equation (3.25) order of y_k and y'_k is equal to the order of y_k^0 and $y_k'^0$ i.e. μ^{-1}.

Therefore, as a result of (3.8), (3.21) and (3.24) we have

$$\{q_i - z_i\} = 0\ (\mu^{-1}). \tag{3.26}$$

Formula (3.26) provides the basis for the applied theory of the gyroscopes.

Remarks: The convergence of rows in (according to) the Poincare method requires that the variables $q_{i1} - z_i$, $\dot{q}_{i1} - \dot{z}_i$ (in paragraph 1) and $x_i - y_i$, $x'_i - y'_i$ (in paragraph 2) are bounded. Therefore, the supposed (anticipated) asymptotic estimations preserve their values infinite time-interval, only if the above referred variables are bounded. Generally in case of an unstable gyroscopic system the estimations obtained hold good at any time if the variable lie in a sufficiently large but bounded region. The remarks are also valid in what follows.

Let us find the values of the positional velocities. By setting up $\mu = \mu^{-2}$ and taking into account that q''_{k1} have the order of μ, we find

$$\{x_i - z_i,\ x'_i - z'_i\} = 0\ (\mu^{-2}),\ \{\dot{x}_i - \dot{z}_i\} = 0\ (\mu^{-1}). \tag{3.27}$$

The variables z_i have the same initial conditions. z_i are determined by the equations.

$$a_{ik}(q^0_\gamma + z_\gamma, t)\ddot{z}_k + \tfrac{1}{2}c_{ik}(q^0_\gamma + z_\gamma, t)\dot{z}_k =$$

$$= -\Omega_i(q\gamma_1 + z_\gamma, \varepsilon^{-1}q'\gamma_1 + \varepsilon z'_\gamma, \varepsilon z', t) +$$

$$+\Omega_i(q\gamma_1, \varepsilon^{-1}q'\gamma_1, t) - a_{ik}(q^0_\gamma + z_\gamma, t)\ddot{q}_{k1}(t). \quad (3.23)$$

If the conditions (a) are satisfied, then q''_{k1} have the zero order in respect of ε and the term containing $q''_{k1}(0)$ may be dropped. When $\varepsilon = 0$, the right hand side of (3.23) vanishes.

It follows from equations (3.22) that the derivatives q_{i1} have the order of ε^{-1} and, therefore, in accordance with (3.8), (3.21) and (3.23) the positional velocities $\{q_i - z_i\} = O(\varepsilon^{-1})$.

Let us discuss as an example a gyroscope with a variable mass and whose equations of motions are given in section § 1.

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + C(\dot{\varphi} + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta = mgl\sin\theta,$$

$$A\frac{d}{dt}(\dot{\psi}\sin^2\theta) + C\frac{d}{dt}\left[(\dot{\varphi}+\dot{\psi}\cos\theta)\cos\theta\right] = K\cos\theta.$$

$$C\frac{d}{dt}(\dot{\varphi}+\dot{\psi}\cos\theta) = K.$$

The moments of inertia A and C, pendulum moment mgl and reaction moment K are known functions of time. Therefore, we get

$$\dot{\varphi} + \dot{\psi}\cos\theta = H + h, \quad h = \int_0^t \frac{K}{C}\,dt.$$

Equations (3.5) assume the form

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + C(H + h)\dot{\psi}\sin\theta = mgl\sin\theta,$$

$$A\dot{\psi}\sin^2\theta + 2A\dot{\theta}\dot{\psi}\sin\theta\cos\theta - C(H + h)\dot{\theta}\sin\theta = 0.$$

In consonance with the applied theory of the gyroscopes we get

$$C(H + h)\dot{\psi} = mgl,\ \theta = 0,\ \dot{\theta} = \theta_0,\ \psi = \int_0^t \frac{mgl}{C(H + h)}\,dt + \psi_0. \qquad (3.29)$$

The slow precession (3.29) upto the terms of the order of H^{-1} determins the solution of exact equations.

Thus it leads us to the following conclusion. If the fast rotating gyroscope of a variable mass deviates from the vertical position through an angle θ_0, then it will produce in variables θ and ψ oscillations whose amplitude is of the order of H^{-1} and frequencies of the order of H about the motion (3.29).

3. Motion of a Gyroscopic System Placed on a Mobile Foundation.

As in paragraph 2, we will first study the motion of a stable system. In order to do this we will study the linearized equations.

Restricting to the terms of the first order of smallness for q_i and $\dot{q}_i$ in equations (3.5) we obtain the equations

$$a_{ik}\ddot{q}_k + (b_{ik} + Hg_{ik})\dot{q}_k + (c_{ik} + Hd_{ik})\,q_k = f_i + HF_i. \qquad (3.30)$$

Here $\|a_{ik}\|$ is a matrix of a positive definite quadratic from. The coefficients in equations (3.30) are explicit functions of time and do not depend upon the parameter H.

Equations (3.7) after the linearization assume the form

$$(\overset{v}{b}_{ik} + H\overset{v}{G}_{ik})\dot{\delta}_k + (\overset{v}{c}_{ik} + H\overset{v}{d}_{ik})\delta_k = \overset{v}{f}_i + HF_i. \qquad (3.31)$$

Here the sign $\overset{v}{\ }$ indicates that the given coefficients differ from those recorded previously.

REMARKS:

1) In the real gyroscopic system the coefficients d_{ik} and F_i may have a component of the angular velocity of rotation of the earth which will be a sufficiently small quantity, the coefficients c_{ik} may have a component of the pendulum moment which in certain gyroscopic systems will be close to H. Therefore, in any concrete system we should study its special features.

2) If the generalized, ordinary and reaction forces are not expressed as explicit functions of time, then the equations (3.30) and (3.31) are obtained only if it is possible to find $\varphi_\nu = \varphi_\nu(\delta_i, q_i, t)$ from the Lagrange's equation of second type for the cyclic coordinates.

As before, assuming that the determination of the matrix of the gyroscopic terms $\|G_{ik}\|$ is different from zero, we consider the solution of equations (3.31) with arbitrary initial conditions $\delta_i^0 = q_i^0$. The variables δ_i have the same order as the initial conditions, i.e., zero in respect of H. The order of q_i is not higher than zero.

Let q_i have order 0' in respect of H. Consider the solution of equations

$$(b_{ik} + H\mathring{c}_{ik})\,\dot{q}_{k1} + (c_{ik} + Hd_{ik})\,q_{k1} = f_i + HF_i \qquad (3.33)$$

with initial conditions $q^o_{k1} = q^o_k$.

To find the variables $x_i = q_{i1} - q_i$, we write the equations (3.31) and (3.32) and get

$$(H^{-1}\,b_{ik} + c_{ik})\,\dot{x}_k + (H^{-1}\,c_{ik} + d_{ik})\,x_k =$$

$$= H^{-1}\left[(\check{b}_{ik} - b_{ik})\,\dot{q}_k + (\check{c}_{ik} - c_{ik})\,q_k + f_i - \check{f}_i\right],$$

from this it is easy to see that

$$\left\{q_{i1} - q_i,\ \dot{q}_{i1} - \dot{q}_i,\ \ddot{q}_{i1} - \ddot{q}_i\right\} = 0\,(H^{-1}). \qquad (3.33)$$

Let $q_i^{(1)}$ characterize the motion of the gyroscopic system (3.30) with initial conditions

$$q_k^{(1)o} = q_k^o,\ \left[b_{ik}\,(0) + Hc_{ik}\,(0)\right]\dot{q}_k^{(1)o} + \left[c_{ik}\,(0) + Hd_{ik}\,(0)\right]q_k^o =$$

$$= f_i\,(0) + HF_i\,(0). \qquad (3.34)$$

In view of (3.30), (3.32) and (3.34), $y_i = q_i^{(1)} - q_{i1}$ represents the particular solution with the zero initial conditions of the system of equations

$$a_{ik}\ddot{y}_k + (b_{ik} + Hc_{ik})\,\dot{y}_k + (c_{ik} + Hd_{ik})\,y_k = -\,a_{ik}\ddot{q}_{k1}. \qquad (3.35)$$

It follows from (3.33) and (3.35) that the variables y_k and $\dot{y}_k$ have the order of O'' which is the maximum of the values of O' and H^{-1}.

The solution of equations (3.30) with arbitrary initial conditions q_i^o, $\dot{q}_i^o$ is $q_i = q_i^{(1)} + q_i^{(2)}$, where $q_i^{(2)}$ is the solution of the homogeneous equations of the system (3.30) and satisfies the initial conditions

$$q_i^{(2)}{}_o = 0, \quad \dot{q}_i^{(2)}{}_o = q_i^o - \dot{q}_i^{(1)}{}_o \tag{3.36}$$

Set up $\tau_2 = Ht$ and symbolizing the derivatives in respect of τ_2, by the sign "11" we get

$$a_{ik}q_k^{(2)''} + (H^{-1}b_{ik} + q_{ik})\, q_k^{(2)'} + H^{-2}\,(C_{ik} + Hd_{ik})\, q_k^{(2)} = 0.$$

Consider a solution satisfying the initial conditions (3.36) of the system of equations:

$$a_{ik}q_{k2}'' + (H^{-1}b_{ik} + g_{ik})\, q_{k2}' = 0. \tag{3.37}$$

The variables q_{k2}' possess the same order as the initial conditions i.e. H^{-1}. $z_k = q_k^{(2)} - q_{k2}$ are obtained as particular solutions with the zero initial conditions of the equations

$$a_{ik}z_k'' + (H^{-1}b_{ik} + g_{ik})\, z_k' + H^{-2}(C_{ik} + Hd_{ik})\, z_k =$$

$$= -H^{-2}\,(C_{ik} + Hd_{ik})\, q_{k2}. \tag{3.38}$$

Solve the equations (3.37) in respect of q'_{k2} and integrate them with respect τ_2. This reveals that q_{k2} has order H^{-1}. In order to find a particular solution of equations (3.38) with the zero initial conditions it is necessary to integrate with respect to τ_2 keeping the range from zero to Ht. Therefore, it follows that $\{z'_k, z_k\} = O(H^{-1})$.

Consequently, we get

$$q_i = g_i + O'', \quad \dot{q}_i = \dot{g}_i + q_i^{(2)} + O'',$$

which serve as the foundation of applied theory of gyroscopes. It may be emphasised that z_k possesses zero order in respect of H and in the present case $\dot{q}_i^{(2)}$ cannot be replaced by the velocities $\dot{q}_{i2}$. Unlike the case of the gyroscopic system placed on a fixed foundation, $\dot{q}_{i1}$ cannot have the order of H^{-1}.

Let as now consider the case when the gyroscopic sytem has finite deviations and the generalized forces are of the order of H (for example pendulum moment). Suppose $\ddot{g}_i$ which obtained by differentiating equations (3.37) have the order in respect of H. We find a solution of equations (3.5) in the form of (3.21). The variables q_{i1} satisfy the equations

$$H(g_{ik}^{(1)} \dot{q}_{k1} + g_{io}^{(1)} + \Omega_i^{(1)} = 0. \qquad (3.39)$$

In this case, q_{i1} cannot be taken as function of the argument $\tau_1 = H^{-1}t$. To find x_i we get equations

$$a_{ik}(q\gamma_1 + x_\gamma, H^{-1}\tau_2)\, x_k'' + g_{ik}(q\gamma_1 + x_\gamma, H^{-1}\tau_2)\, x_k' +$$

$$+ H^{-2}\Big[\Omega_i(q\gamma_1 + x_\gamma, \dot{q}_{\gamma 1} + Hx'_\gamma, H^{-1}\tau_2) - \Omega_i(q\gamma_1, \dot{q}_{\gamma 1}, H^{-1}\tau_2) +$$

$$+ \psi_i(H^{-1}, \mu, x_\gamma, \tau_2) = 0,$$

where

$$\psi_i = a_{ik}(q\gamma_1 + x_\gamma, H^{-1}\tau_2)\, q_{k1} + H^{-1}\Big[g_{ik}(q\gamma_1 + x_\gamma, H^{-1}\tau_2) -$$

$$- q_{ik}(q\gamma_1, H^{-1}\tau_2)\Big]\dot{q}_{k1} + H^{-1}\Big[g_{io}(q\gamma_1 + x_\gamma, H^{-1}\tau_2) - g_{io}(q\gamma_1, H^{-1}\tau_2)$$

when ξ_k has the zero order, we have

$$O\Big[\Omega_i(\xi_j, \eta_j, t)\Big] \leqslant O'', \quad O'' = \max\Big\{H^2, O(\eta^\sigma) \quad (\sigma = 0, 1, 2).$$

This will explain the interesting cases of the gyroscopic systems.

The functions ψ_i depend upon both H^{-1} and μ and vanish with them. In accordance with the method of small parameter due to Poincre we have

$$O\Big\{x_i - y_i, x_i' - y_i'\Big\} = O'. \tag{3.40}$$

where $O' = \max\Big(H^{-1}, \quad\Big)$.

The vapiables y_i are given by the equations

$$a_{ik}(q\gamma_1 + y_\gamma, H^{-1}\tau_2)\, y_k'' + g_{ik}(q\gamma_1 + y_\gamma, H^{-1}\tau_2)\, y_k' +$$

$$+ H^{-2}\Big[\Omega_i(q\gamma_1 + y_\gamma, \dot{q}_{\gamma 1} + Hy'_\gamma, H^{-1}\tau_2) - \psi_i(q\gamma_1, \dot{q}_{\gamma 1}, H^{-1}\tau_2)\Big] = 0. \tag{3.41}$$

From equations (3.3) it will be seen that y_i and y_i' have the order of the initial conditions i.e. H^{-1}. (3.8) and (3.40) show that

$$0\left\{q_i - g_i\right\} = 0'. \qquad (3.42)$$

Formula (3.42) will substantiate the equations of applied theory of the gyroscopes on mobile formation.

4. Effects of Change on the Mass of Yarn on the Vibration of the Spindle.

The modern fast spindles can be regarded as the gyroscopes and [22, 31, 41, 42]. The two supports of the stem of the spindle are installed in the hub which hangs from the cavity by elastic bends. As the number of revolutions increase the stem remains tight for some time, therefore, we will carry out the investigations, both when the stem is tight and elastic.

1. Fundamental dynamical characteristics of the cylenderical and canonical yarns. We will first establish the dependence of the fundamental dynamical characteristics of the cylenderical yarn on time. To fix the ideas, we will consider the winding of the thread.

Let γ - be the density of the thread,

d - its diameter,

σ - mass per unit length of the thread,

ω - angular velocity of the rotation of stem,

ω_1 - angular orbital driving velocity,
ω' - angular velocity of winding,
ρ - radius of yarn,
T - time taken in winding one strature,
m - mass of the spindle.

We will use the index to denote the zero values of the variables at the time when we begin to take account of the vibrations and which we will suppose to be zero. We thus get

$$\sigma = \gamma \frac{\pi d^2}{4}, \quad \omega' = \omega - \omega_1, \quad \dot{m} = \sigma \omega' \rho, \quad = \rho_0 + \frac{d}{T} t$$

It follows that

$$m = m_0 + \sigma \omega' \rho_0 t + \frac{\sigma \omega' d}{T} \cdot \frac{t^2}{2}.$$

The last term has the order of d^3 and this can be neglected. Taking into account the small parameter $\mu = \sigma \omega'$ and denoting $m' = \rho_0$, we get

$$m = m_0 + \mu m' t. \tag{4.1}$$

The derivative in respect of time of the moment of inertia of the spindle about its own axis is given by

$$A = (\rho_0 + \frac{d}{T} t)^3 \sigma \omega'.$$

Therefore, with the points before the term d^2

$$A = A_0 + \mu A' t, \quad A' = \rho_0^3. \tag{4.2}$$

and also the derivative in respect of time of the moment of the inertia about the tranverse axis is given by

$$B = (\xi^2 + \eta^2) \sigma \omega' \rho,$$

where $\xi = L + f(t + t')$ is the projection of the radius vector of the surface point of the yarn on its axis, L is the distance of a fixed point from the center of the yarn; f is periodic function with the angular frequency π/T oscillating about zero, amplitude of the oscillation of f is equal to half the length of the yarn (1/2H); t' is determined from the condition

$$\xi^0 = L + f(t');$$

$\eta = \rho \cos(\omega' t + \alpha)$ represents the projection of the radius vector of the surface point of the yarn on the second transverse axis; is the starting phase from here; we thus find

$$B = B_0 + \mu \rho_0 \left(L^2 t + 2L \int_0^t f dt + \int_0^t f^2 dt + \frac{1}{2} \rho_0^2 t + \frac{\rho_0^2 \sin 2(\omega' t + \alpha)}{4\omega'}\right).$$

Neglecting the second and the fifth terms in the brackets and averaging (approximating) the second integral, we get

$$B = B_0 + \mu B' t, \quad B' = \rho_0 \left(L^2 + \frac{1}{2} \rho_0^2 + \frac{1}{8} H^2\right). \qquad (4.3)$$

In case of cannorical yarn

$$\rho = \rho_0 - \frac{r_2 - r_1}{H} f(t + t') + \frac{d}{T} t,$$

where r_2 is lower and r_1 is upper radii of the yarn at the starting time (beginning); $\rho_0 = \frac{1}{2}(r_1 + r_2)$. Repeating the earlier process and taking into account that $\omega' \gg \frac{\pi}{T}$, for the canonical yarn, we have

$$m' = \rho_0,\ A' = \rho_0 \left[\rho_0^2 + \frac{3}{8}(r_2 - r_1)^2\right],$$

$$B' = \rho_0 \left[L^2 + \frac{1}{2}\rho_0^2 + \frac{1}{8}H^2 + \frac{3}{8}(r_2 - r_1)^2\right] - \frac{1}{4}LH(r_2 - r_1). \tag{4.4}$$

2. Vibration of the rigid stem of the spindle with a variable yarn. If the tension (pull) of the thread and the reaction force caused by the addition of mass are not taken into account, then the equations of the oscillation of the rigid stem are analogous to the equations of the empty stem or when the stem is full of yarn (31).

$$(B_0 + \mu B't)\ddot{y} + (A_0 + \mu A't)\omega\dot{z} + (\nu_0 + \mu\nu't)y =$$

$$= -(m_0 + \mu m't)\omega^2 lph\cos\omega t, \tag{4.5}$$

$$(B_0 + \mu B't)\ddot{z} - (A_0 + \mu A't)\omega\dot{y} + (\nu_0 + \mu\nu't)z =$$

$$= -(m_0 + \mu m't)\omega^2 lph\sin\omega t.$$

Let (x, y, z) be a fixed system whose origin coincides with a fixed point of the stem, and its vertical axis is regarded as x-axis. y and z coordinates are below the changing elastic support whose distance from the fixed point is l; h denotes the projection of the radius vector of the center of gravity of the spindle on its own axis; p - eccentricity. For convenience, we suppose that the change of mass does not affect the change in the values of l and p. This does not make substantial changes in the final result. In equations (4.5)

$$\nu_0 = Cl^2 - m_0 gh,\quad \nu' = -m'gh,$$

where C is a constant. In the case of modern spindles we can assume $B_o \gg A_o$.

Pull or the tension of the thread will be characterized as a periodic function with period 2T. The point of application of this force turns around the x-axis with the angular velocity ω'. The moments of the forces of the tension about y or z axis will change sign as it is wound in one or the other direction. Therefore, these moments will be of the form

$$\begin{aligned} &\left[F(t + t'') + \mu t F'(t + t'')\right] \sin(\omega' t + \varepsilon), \\ &\left[F(t + t'') + \mu t F'(t + t'')\right] \cos(\omega' t + \varepsilon), \end{aligned} \tag{4.6}$$

where F and F' are odd periodic functions of the angular frequency $\frac{\pi}{T}$. Time t'' and phase are found from the initial conditions.

Reaction force according to Mesherky's formula is equal to

$$\frac{dm}{dt}(\bar{u} - \bar{v}) \quad \left(\frac{dm}{dt} = \sigma v,\ v = \omega' \rho,\ \bar{u} = \bar{v} \pm v \operatorname{tg}\beta \bar{k},\ \bar{k} \perp \bar{v},\ |\bar{k}| = 1\right).$$

where $\bar{u}$ is the velocity of the particles of the thread before (till) they join the yarn; $\bar{v}$ is peripheral velocity of the yarn; β is angle of the winding (signs change periodically in the interval of time T sign is plus and then in the next interval of time T sign becomes minus). The moments of reaction force about y and z axes are respectively equal to

$$+\mu\omega'^2\rho^2 \operatorname{tg}\beta \sin(\omega' t + \varepsilon),\ \pm \mu\omega'^2\rho^2 \operatorname{tg}\beta \cos(\omega' t + \varepsilon).$$

The moments of the reaction force have a periodic character like the moments of the force of tension. Therefore, the moments of the two types of the forces are set out in the formula (4.6).

The odd periodic functions F and F' may be replaced by a trigonometric series without sacrificing the accuracy. To fix the ideas, we replace them by the first hormonics. Then in place of (4.6) we have

$$(a + \alpha bt) \sin \frac{\pi}{T} (t + t'') \sin (\omega' t + \varepsilon),$$

$$(a + \alpha bt) \sin \frac{\pi}{T} (t + t'') \cos (\omega' t + \varepsilon).$$

After making the transformation, the moment about y axis is

$$\frac{1}{2} (a + \alpha bt) \left\{ \cos \left[\left(\frac{\pi}{T} - \omega' \right) t + \frac{\pi}{T} t'' - \varepsilon \right] - \right.$$

$$\left. - \cos \left[\left(\omega' + \frac{\pi}{T} \right) t + \left(\varepsilon + \frac{\pi}{T} t'' \right) \right] \right\} . \qquad (4.7)$$

Moment about z - axis is

$$\frac{1}{2} (a + \alpha bt) \left\{ \sin \left[\left(\frac{\pi}{T} - \omega' \right) t + \frac{\pi}{T} t'' - \varepsilon \right] + \right.$$

$$\left. + \sin \left[\left(\omega' + \frac{\pi}{T} \right) t + \left(\varepsilon + \frac{\pi}{T} t'' \right) \right] \right\} . \qquad (4.8)$$

Let us add moment (4.7) to the first equation of the system (4.5) and the moment (4.8) to the second equation.

The intrinsic vibrations of the spindle die out on account of damping which is not taken into consideration while writing out the equations (4.5). The forced oscillations of the spindle $\xi = y + iz$ are determined by a particular solution of the system (4.5).

Assuming $\zeta = y + iz$, we get

$$(B_0 + \mu B't)\ddot{\zeta} - i\omega(A_0 + \mu A't)\dot{\zeta} + (\nu_0 - \mu\nu't)\zeta =$$

$$= -(m_0 + \mu m't)\omega^2 lphe^{i\omega t}. \qquad (4.9)$$

The dynamical characteristics of the spindle can be represented as polynomials of the degrees $\mu' = \sqrt{\mu}$, and therefore, the motion of the stem is anticipated in the form

$$\zeta = \zeta_0 + \mu'\zeta_1 + \mu'^2\zeta_2 + \ldots \qquad (4.10)$$

Since the equations (4.5) are worked out with accuracy upto μ, and this puts a restriction on the solution (4.10).

ζ_0 is a particular solution of the equation

$$B_0\ddot{\zeta}_0 - i\omega A_0\dot{\zeta}_0 + \nu_0\zeta_0 = -m_0\omega^2 lphe^{i\omega t}. \qquad (4.11)$$

This gives

$$\zeta_0 = a_1 e^{i\omega t},\quad a_1 = \frac{m_0\omega^2 lph}{(B_0 - A_0)\omega^2 - \nu_0}$$

The first term, obtained as particular solution of equation (4.11) when its right side is zero, is equal to zero. Further we have

$$B_0\ddot{\zeta}_2 - i\omega A_0\dot{\zeta}_2 + \nu_0\zeta_2 = \left\{-m'\omega^2 lph + \left[(B' - A')\omega^2 - \nu'\right] a_1\right\} te^{i\omega t}. \qquad (4.12)$$

This yields $\zeta_2 = (a_2 t + ia_3)\, e^{i\omega t}$, where

$$a_2 = \frac{m' \omega^2 l_{ph} - \left[(B' - A')\omega^2 - \nu\right] a_1}{(B_o - A_o)\omega^2 - \nu_o}, \quad a_3 = \frac{(2B_o - A_o)\omega a_2}{(B_o - A_o)\omega^2 - \nu_o}.$$

Subsequent approximation may be obtained in the same way if we put the exact formulae of A and B in equation (4.9). By transforming to the variables y and z we get

$$\begin{aligned} y &= (a_1 + \mu a_2 t)\cos\omega t - \mu a_3 \sin\omega t, \\ z &= (a_1 + \mu a_2 t)\sin\omega t + \mu a_3 \cos\omega t. \end{aligned} \tag{4.13}$$

The forced oscillations due to the tension and reaction force will be found in the same way. By proceeding from (4.7) - (4.10) for the zero approximation in this case we have

$$B_o\zeta_o - i\omega A_o \zeta_o + \nu_o \zeta_o = \frac{1}{2} a \left\{\exp\left[-i(\Omega_1 t + \varepsilon_1)\right] - \exp\left[-i(\Omega_2 t + \varepsilon_2)\right]\right\}. \tag{4.14}$$

where

$$\Omega_1 = \omega' - \frac{\pi}{T}, \quad \Omega_2 = \omega' + \frac{\pi}{T}, \quad \varepsilon_1 = \varepsilon - \frac{\pi}{T} t'', \quad \varepsilon_2 = \varepsilon + \frac{\pi}{T} t''.$$

From (4.14), we get,

$$\zeta_o = a_{11} \exp\left[-i(\Omega_1 t + \varepsilon_1)\right] + a_{12} \exp\left[-i(\Omega_2 t + \varepsilon_2)\right],$$

where

$$a_{11} = -\frac{a}{2(B_o\Omega_1^2 + A_o\omega\Omega_1 - \nu_o)}, \quad a_{12} = \frac{a}{2(B_o\Omega_2^2 + A_o\omega\Omega_2 - \nu_o)}$$

We have $\xi_1 = 0$, and to find ξ_2 we have the equations

$$B_0\ddot{\xi}_2 - i\omega A_0\dot{\xi}_2 - \nu_0\xi_2 = \left[\frac{b}{2} + (B'\Omega_1^2 + A'\omega\Omega_1 - \nu')\, a_{11}\right] t \exp\left[-i\,X\right.$$

$$\left.X\,(\Omega_1 t + \varepsilon_1)\right] - \left[\frac{1}{2}b + (B'\Omega_2^2 + A'\omega\Omega_2 - \nu')a_{12}\right] t \exp\left[-i(\Omega_2 t + \varepsilon_2)\right]$$

This yields

$$\xi_2 = (a_{21}t + i\Omega\, a_{31}) \exp\left[-i\,(\Omega_1 t + \varepsilon_1)\right] +$$

$$+ (a_{22}t + i\Omega_2 a_{32}) \exp\left[-i\,(\Omega_2 t + \varepsilon_2)\right],$$

where

$$a_{2k} = (-1)\,\frac{\frac{1}{2}b + (B'\Omega_k^2 + A'\omega\Omega_k - \nu')\, a_{1k}}{B_0\Omega_k^2 + A_0\omega\Omega_k - \nu_0}, \quad a_{3k} = \frac{(2B_0 - A_0)\Omega_k a_{2k}}{B_0\Omega_k^2 + A_0\omega\Omega_k - \nu_0}$$

$$(k = 1, 2).$$

Writing in y and z variables, for the forced oscillations due to the tension and the reaction force we get

$$y = \sum_{k=1}^{2}\left[(a_{1k} + \mu a_{2k}t)\cos(\Omega_k t + \varepsilon_k) + \mu a_{3k}\sin(\Omega_k t + \varepsilon_k)\right],$$

$$z = \sum_{k=1}^{2}\left[-(a_{1k} + \mu a_{2k}t)\sin(\Omega_k t + \varepsilon_k) + \mu a_{3k}\cos(\Omega_k t + \varepsilon_k)\right].$$

When the motion is restricted in the neighborhood of the chosen initial moment, the critical angular velocities $\check{\omega}$ of hard stem are given by the relation

$$(B_0 - A_0)\check{\omega}^2 - \nu_0 = 0, \quad B_0\Omega_k^2 + A_0\check{\omega}\Omega_k - \nu_0 = 0.$$

In the course of motion in the neighborhood of another moment, the moments of inertia and mass of the spindle correspondingly change.

Therefore, the change in mass of the yarn does not substantially influence the first critical velocities $\check{\omega}_1$ and $\check{\omega}_2$. A smearing of the critical velocities takes place as a result of change of mass. Since $\Omega_1 \approx \Omega_2 \approx \omega'$, the spindle has two critical velocities of hard stem which satisfy the following inequalities.

$$\min \sqrt{\frac{\nu}{B - A}} \leqslant \check{\omega}_1 \leqslant \max \sqrt{\frac{\nu}{B - A}}, \min \frac{\nu - B\omega'^2}{A\omega'} \leqslant$$

$$\leqslant \check{\omega}_2 \leqslant \max \frac{\nu - B\omega'^2}{A\omega'}$$

It may be observed that the amplitudes of the oscillations $\mu a_2 t$, $\mu a_{2k} t$, and μa_3, μa_{3k} caused by the process of change of mass when $\omega \to \check{\omega}$ increase faster than the corresponding amplitudes a_1 or a_{1k} because a square and cube of one of the expressions $(B_o - A_o)\omega^2 - \nu_o$, $B_o\Omega_k^2 + A_o\omega\Omega_k - \nu_o$ appear as denominator of a_2, a_{2k}, a_3, a_{3k}. In order to find the dynamical characteristics, for example, reactions of the supports under the angular velocities close to $\check{\omega}$, we will study the oscillations produced as a result of the process of change of mass.

In the case of resonance when $(B_o - A_o)\omega^2 - \nu_o = 0$, the particular solution of the equations (4.11) has the following form:

$$\zeta_o = ib_1 te^{i\omega t}, \quad b_1 = \frac{m_o \omega lph}{2B_o - A_o}$$

The particular solution of equations (4.12) is obtained in the form

$$\zeta_2 = (b_2 t^3 + ib_3 t^2 + b_4 t)\, e^{i\omega t},$$

in which

$$b_2 = \frac{\left[(B' - A')\omega^2 - \nu'\right] b_1}{3\,(2B_0 - A_0)}, \quad b_3 = \frac{-m'\omega l p h + (2B' - A')b_1}{2\,(2B_0 - A_0)},$$

$$b_4 = -\frac{2B_0 b_3}{\omega(2B_0 - A_0)}.$$

In case of resonance in terms of y and z we get

$$y = -(b_1 t + \omega b_3 t^2)\sin\omega t + \mu(b_2 t^3 + b_4 t)\cos\omega t,$$

$$z = (b_1 t + \omega b_3 t^2)\cos\omega t + \mu(b_2 t^3 + b_4)\sin\omega t. \qquad (4.15)$$

Consequently, under resonance the amplitude of the terms increase with t^2 and t^3. The amplitude of the ordinary oscillations is proportional to time.

In modern spindles ω_2 is ten times more than ω_1 and with further increase in the operating velocity of the stem ω_2 will come in the range of the operating velocities. Angalogously, investigations about resonance are carried out when $B_0\Omega_k^2 + A_0\omega\Omega_k - \nu_0 = 0$. The resonance oscillations will be described by equations (4.15) in which ω is changed for Ω_k and $t + \frac{\varepsilon_k}{\Omega_k}$ replace t.

3. Oscillations of the elastic stem of the spindles with a variable yarn. At a certain angular velocity, the stems of the spindle gets rigid. We will study the vibrations of the flexible stem owing to the change of mass of the yarn.

Denoting by $u_q(x, t)$ and $v_q(x, t)$ the sagging in the directions of y and z axes, we find that the complex sagging on the q-th sector of

the stem, $\omega_q(x, t) = u_q(x, t) + iv_q(x, t)$, will satisfy the differential equation [41, 42]

$$\frac{\partial^4 \omega_q(x, t)}{\partial x^4} + k_q^4 \frac{\partial^2 \omega_q(x, t)}{\partial t^2} = 0, \qquad (4.16)$$

in which $k_q^4 = \frac{m_q}{EI_q}$. m_q and EI_q - are the mass and per unit length and hardness on the curve (bend) of q-th sector of the stem. The stem is divided into parts according to

suppose the stem is divided into k_q parts. Then the solution of equations (4.16) must satisfy 4 n boundary complex conditions [41, 42].

Proceeding from (4.1) - (4.4), the boundary conditions may be written in the form

$$\sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} (P^{0\nu\gamma}_{qk} + \mu P'^{\nu\gamma}_{qk}) \left(\frac{\partial^{\nu+\gamma}\omega_q + 1}{\partial x^\nu \partial t^\gamma}\right)_{x=lq} \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} (Q^{0\nu\gamma}_{qk} - \mu Q'^{\nu\gamma}_{qk})$$

$$\times \left(\frac{\partial^\nu + \gamma \omega_q}{\partial x^\nu \partial t^\gamma}\right)_{x=0} = (R^0_{qk} + \mu R'_{qk} t)\, e^{i\omega t} \quad (k = 1, 2, 3, 4;\ q = 1, 2, \ldots, n). \qquad (4.17)$$

$P^{0\nu\gamma}_{qk}$, $P'^{\nu\gamma}_{qk}$, $Q^{0\nu\gamma}_{qk}$, $Q'^{\nu\gamma}_{qk}$, R^0_{qk}, R'_{qk} are constants

l_q - length of q-th parts of stem.

The non-homogeneity of the conditions (4.17) is due to static and dynamic disbalance. If we take into account the tension of the thread and the reaction force, then on the right side of equations (4.17) we will add the terms

$$(R_{qk}^{01} + \mu R_{qk}^{\prime 1} t) \exp\left[-i\left(\omega' - \frac{\pi}{T}\right)t\right] +$$

$$+ R_{qk}^{02} + R_{qk}^{\prime 2} t) \exp\left[-i\left(\omega + \frac{\pi}{T}\right)t\right].$$

Since the intrinsic oscillations of the flexible stem subside, we will consider only the forced oscillations. These consist of the oscillations arising out of the disbalance on the one hand and those produced by the tension and the reaction force, on the other hand. Let us consider the forced oscillations due to disbalance. The forced oscillations due to tension and reaction force will be dealt with in the same manner.

The forced oscillations due to disbalance are determined by the equation (4.16) and the condition (4.17). Suppose

$$w_q = \sum_{s=1}^{4} (a_{qs}^1 + \mu a_{qs}^2 t + \mu a_{qs}^3) K_s(-\sqrt{\omega}\, k_q x) e^{i\omega t}, \quad (4.18)$$

in which a_{qs}^1, a_{qs}^2, a_{qs}^3 are real constants, $K_s(y)$ - represent the Carilova functions satisfying the equation $K_s^{(4)}(y) - K_s(y) = 0$.

Put the expression for w_q in (4.17)

$$\sum_{s=1}^{4}\left\{\sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} (P_{qk}^{0\nu\gamma} + \mu P_{qk}^{\prime\nu\gamma} t)\left[(a_{q+1,s}^1 + \mu a_{q+1,s}^2 + \mu a_{q+1,s}^3)(i\omega)^\gamma + \right.\right.$$

$$\left. + \mu(2i\omega)^{\gamma-1} a_{q+1,s}^2\right] K_s^{(\nu)}(-\sqrt{\omega}\, k_{q+1} l_q) +$$

$$+ \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} (Q_{qk}^{0\nu\gamma} + \mu Q_{qk}^{\prime\nu\gamma} t)\left[(a_{qs}^1 + \mu a_{qs}^2 t + \mu a_{qs}^3)(i\omega)^\gamma + \right.$$

$$\left.\left. + \mu(2i\omega)^{\gamma-1} a_{qs}^2\right] K_s^{(\nu)}(0)\right\} = R_{qk}^0 + \mu R'_{qk} t. \quad (4.19)$$

Assume $\mu = 0$, then the constants will be found from a system containing $4n$ equations

$$\sum_{s=1}^{4} (A_{qk}^{s}\, a_{q+1,s}^{1} + B_{qk}^{s} a_{qs}^{1}) = R_{qk}^{0} \quad (k = 1, 2, 3, 4;\ q = 1, 2, \ldots, n), \tag{4.20}$$

$$A_{qk}^{s} = \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} P_{qk}^{0\nu\gamma}(i\omega)\ K_{s}^{(\nu)}(\sqrt{\omega}\, k_{q+1}, l_q),$$

$$B_{qk}^{s} = \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} Q_{qk}^{0\nu\gamma}(i\omega)^{\gamma} K_{s}^{(\nu)}(0).$$

The constants a_{qs}^{1} do not exist when the determinant formed from t the coefficients of these constants is zero. The values of ω which satisfy the equation $\Delta = 0$ will be regarded as critical of the second and higher order for the time t. By Cramer's rule, a_{qs}^{1} are obtained as fractions whose denominator is Δ. Equating the coefficients of μt on both the sides of equation (4.19) we get the system

$$\sum_{s=1}^{4} (A_{qk}^{s}\, a_{q+1,s}^{2} + B_{qk}^{s}\, a_{qs}^{2}) =$$

$$= R_{qk}^{1} - \sum_{s=1}^{4} a_{qs}^{1} \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} P_{qk}^{1\nu\gamma}(i\omega)^{\gamma} K_{s}^{(\nu)}(\sqrt{\omega}\, k_{q+1} l_q) -$$

$$- \sum_{s=1}^{4} a_{qs}^{1} \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} Q_{qk}^{1\nu\gamma}(i\omega)\ K_{s}^{(\nu)}(0). \tag{4.21}$$

The $4n$ constants a_{qs}^{2} will be determined from the $4n$ equations (4.21). The determinant of the system is again Δ. Therefore, the constants a_{qs}^{2} will be of the order of Δ^{-2} for the angular velocities approaching K .

Equating free terms of the order of on both sides of equation (4.19), we obtain the system

$$\sum_{s=1}^{4} (A_{qk}^{s} a_{q+1,s}^{3} + B_{qk}^{s} a_{qs}^{3}) = - \sum_{s=1}^{4} \left[a_{q+1,s}^{2} \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} p_{qk}^{0\nu\gamma} (2i\omega)^{\gamma-1} \times \right.$$

$$\left. \times K_{s}^{(\nu)} (-\sqrt{\omega}\, k_{q+1}\, l_{q}) - a_{qs}^{2} \sum_{\substack{0 \le \nu+\gamma \le 3 \\ 0 \le \gamma \le 2}} Q_{qk}^{0\nu\gamma} (2i\omega)^{\gamma-1} K_{s}^{(\nu)}(0) \right]. \qquad (4.22)$$

Therefore, for the angular velocities which are close to the critical velocity $\check{\omega}$, the constants a_{qs}^{3} have the order of Δ^{-3}.

Using (4.18) in equation (4.16) we get

$$\sum_{s=1}^{4} \left[(a_{qs}^{1} + \mu a_{qs}^{2} t + \mu a_{qs}^{3} \omega^{2} k_{q}^{4}) K_{s}^{(4)} - \right.$$

$$\left. - \omega^{2} k_{q}^{4} \left(a_{qs}^{1} + \mu a_{qs}^{2} t + \mu a_{qs}^{3} + 2i\mu \frac{a_{qs}^{2}}{\omega}\right) K_{s} \right] = 0. \qquad (4.23)$$

When the angular velocities are large, the term $2i\omega^{-1} a_{qs}^{2}$ compared with $a_{qs}^{2} t$ is negligibly small. It the angular velocities approach $K\check{\omega}$, then the term $2i\omega^{-1} a_{qs}^{2}$ compared with a_{qs}^{3} is again negligibly small.

Consequently we may think that the expression (4.18) will satisfy the equation (4.16) and the boundary conditions (4.17) upto μ. The effect of the change of mass in the yarn and the tension of the thread on the vibration of the spindle with an elastic stem in similar to that on the spindle with a rigid stem.

CHAPTER - VII

Some aspects of the optimal motion of a body of a variable mass in the central field.

The investigation of the motion of a body of a variable mass has an important application. The optimal trajectories of the motion of a body in this and other contexts have paramount significance. However, the theory of the optimal trajectories in the central field which was first discussed by V. Homan (112) and D.F. Okhotzimskii (76) and many other scientists is beyond the scope of this book. We will confine ourselves to studying the effect of the carioliso force on the optimal direction and the use of the sectors of trajectory with the changing reaction acceleration.

It must be mentioned that these questions can be resolved with reference to a model proble of the optimal transition of a body on circular trajectory. Therefore, the proposed (expected) solution should not be regarded as final and complete. The obtained results have a theoretical interest in the analytical mechanism of a variable mass.

In section § 1 we had pointed out the conditions in which Meshaky's equation can be used to describe the motion of the center of the mass of a body, i.e., the conditions in which it is possible

to study the per second consumption (discharge) of a relative m momentum and a relative kinetic moment together with the principal vector and the principal moment of reaction forces.

The results of this investigation will be applicable to the study of any kind of the motion of a body of a variable mass. The inclusion of this study in the present chapter underlines the significane of the motion of a body in the central field as a fundamental problem of the dynamics of a body of a variable mass. In this connection, it is important to obtain the differential equations of the motion in a simple and convenient form.

Section §2 and 3 deal with coplanar and non-coplanar problems of the optimal motion of a body, if the initial position is fixed, particularly in any of the points of the circular trajectory around the attracting center.

1. Replacement of the Principal Vector and the Principal Moment of the Reaction Forces by the Per Second Consumption (Discharge) of the Relative Momentum and the Relative Kinetic Moment.

1. General formulae for the principal vector and the principal moment of reaction forces acting on a body of a variable mass. As proved in Chapter - V, if we take into account the internal motion of particles in a body, then the principal vector and the principal moment of reaction force can be expressed as

$$\mathfrak{R} = \mathfrak{Q}_1^r - \frac{d'\mathfrak{Q}^r}{dt} - \mathfrak{Q}_2^r + \mathfrak{J}. \qquad (1.1)$$

$$\mathfrak{N} = \mathfrak{I}_1^r - \frac{d'\mathfrak{I}^r}{dt} - \mathfrak{I}_2^r + \mathfrak{H}. \qquad (1.2)$$

$\mathfrak{Q}_1^r$ and $\mathfrak{I}_1^r$ - denote the per second inward relative momentum and relative kinetic moment through the surface of the reactive body; $\mathfrak{Q}_2^r$ and $\mathfrak{I}_2^r$ - denote the per second outward relative momentum and relative kinetic moment through the same surface. $\mathfrak{Q}^r$ and $\mathfrak{I}^r$ - denote the relative momentum and kinetic moment of the particles present in the body. Formulea (1.1) and (1.2) involve their derivatives also.

The principal vector and the principal moment of carioliso forces, as established in chapter V, are given by

$$\mathfrak{J} = 2\left(\mathfrak{Q}_1^e - \frac{d\mathfrak{Q}}{dt} + \frac{d^*\mathfrak{Q}}{dt} - \mathfrak{Q}_2^e\right), \qquad (1.3)$$

$$\mathfrak{H} = \mathfrak{I}_1^e - \omega \times \mathfrak{I}^r - \frac{d\mathfrak{l}}{dt} + \frac{d^*\mathfrak{I}}{dt} - \mathfrak{I}_2^e. \qquad (1.4)$$

In (1.3) and (1.4), $\mathfrak{Q}_1^e$, $\mathfrak{I}_1^e$ and $\mathfrak{Q}_2^e$, $\mathfrak{I}_2^e$ - denote the per second inward and outward transport momentum and the transport kinetic moment through the surface of the body. ω si the angular velocity of the rotation of body in respect of a fixed space.

The formulae (1.2) and (1.4) are valid when the moments are calculated about any point of the rigid system connected with the body. When it is so, the moments $\mathfrak{I}_1^e$ and $\mathfrak{I}_2^e$ are computed about a system of axes moving translationally with the only point which is regarded as the pole.

2. Valuations of the principal vector and the principal moment of reaction forces [73]. As an example we will consider the motion produced by free liquid. But the formulae (1.1) - (1.4) enable us to carry out similar investigations and obtain analogous results for any type of the fuel (motor). Since the motor operates without the suction of the air, we have $\bar{Q}_1^r = \bar{Q}_1^e = 0$, $\bar{I}_1^r = \bar{I}_1^e = 0$. Denoting the per second discharge of the gas by μ and the relative (average) velocity of the discharge by $\bar{u}_r$, we have

$$\bar{Q}_2^r = \mu \bar{u}_r. \tag{1.5}$$

Taking into consideration the pressure of exhausted particles of gas results in a change of u_r in this formula for an effective exhaust velocity, that is equal to:

$$u = u_r + pS_c/\mu . \tag{1.6}$$

Here p - is the difference between the near pressure of the gases in the output cross section (outlet) and the pressure of the surrounding medium. S_c - is the area of the cross section of the outlet.

The per second consumption (discharge) of the relative kinetic moment is written as

$$\bar{I}_2^r = -\mu u_r \bar{\rho}_c \times \bar{I},$$

Where $\bar{\rho}_c$ - is the radius vector of the middle point of the outlet cross section in respect of the pole; $\bar{I}$ - is the basis vector of the longitudinal axis of the body directed from the outlet cross section to the place of the fuel.

We divide the space inside the body in the direction of the longitudinal axis which is occupied by the gases into the parts of the length $d\rho = v^r dt$, where v^r - is the average relative velocity of the gas in the cross section. The mass of the gas in a subdivision is equal to $\dot{\mu}(t+\tau)d\tau$. τ denotes the time taken by the gas to move from the subdivision to the outlet cross section of the nozzle. Denote by the derivative of the per second consumption of the mass in respect of time and let ρ denote the length of the space occupied by the gas. Then the relative momentum and relative kinetic moment are recorded as

$$\bar{Q}^r = \dot{\mu}\rho\bar{l} + 0\left(\frac{\dot{\mu}\rho^2}{u_r}\bar{i}\right), \qquad (1.7)$$

$$\bar{K}^r = -\dot{\mu}\ \bar{r}_c \times \bar{l} + 0\left(\frac{\ddot{\mu}\rho^2}{u_r}\ \bar{r}_c \times \bar{i}\right). \qquad (1.8)$$

In this case, the expression of the principal vector of the carioliso forces (1.3) is transformed into

$$\bar{J} = 2\dot{\mu}\rho(\bar{v}_T^e - \bar{v}_c^e) + 0\left(\dot{\mu}\omega\frac{\rho^2}{u_r}\right),$$

where $\bar{v}_T^e$ - is the velocity of that point of the body from which the fuel drops, and $\bar{v}_c^e$ - is the velocity of the middle point of the outlet cross section.

According to Euler's formula for the velocity, we have

$$\bar{J} = 2\dot{\mu}\rho(\bar{\omega} \times \bar{l}) + 0\left(\dot{\mu}\omega\frac{\rho^2}{u_r}\right). \qquad (1.9)$$

Formula (1.4) assumes the form

$$\bar{R} = -\dot{\bar{\omega}} \times \bar{I}^r - \dot{\mu}\left[\bar{\rho}_c \times (\bar{\omega} \times \bar{\rho}_c) - \bar{\rho}_T \times (\bar{\omega} \times \bar{\rho}_T)\right] + O\left(\dot{\mu}\omega \frac{\rho^3}{u_r}\right),$$

where $\bar{\rho}_T$ - is the radius vector of the middle point from which the fuel drops with reference to the pole. Since $\bar{\rho}_T = \bar{\rho}_c + \rho\bar{i}$, then

$$\bar{R} = -\dot{\bar{\omega}} \times \bar{I}^r + \dot{\mu}\rho\left[\bar{\rho}_c \times (\bar{\omega} \times \bar{i}) + \bar{i} \times (\bar{\omega} \times \bar{\rho}_c) + \right.$$

$$\left. + \rho\bar{i} \times (\bar{\omega} \times \bar{i})\right] + O\left(\dot{\mu}\omega \frac{\rho^3}{ur}\right). \qquad (1.10)$$

From (1.5), (1.7) and (1.9) we find that the principal vector of the reaction forces (1.1) is equivalent to

$$\bar{R} = -\bar{Q}_2^r - \ddot{\mu}\rho\bar{i} + 2\dot{\mu}\rho\bar{\omega} \times \bar{i} + O\left(\ddot{\mu}\omega \frac{\rho^2}{u_r}, \dot{\mu}\dot{\omega} \frac{\rho^3}{u_r}\right).$$

(1.11)

Consider the axes x, y, z, firmly connected with the body. Let the origin of the system be at the pole, and let the x axis along the longitudinal axis of the body and y and z be the arbitrary axes and directed from the output cross section towards the input of fuel, y and z be the arbitary axes.

Planning for a vector equation (1.11) we obtain:

$$R_x = Q_2^r\left[1 + \frac{\ddot{\mu}}{\dot{\mu}} \cdot \frac{\rho}{u_r} + O\left(\frac{\ddot{\mu}}{\dot{\mu}}\omega \frac{\rho^2}{u_r^2}, \frac{\dot{\omega}}{\omega} \cdot \frac{\rho^2}{u_r^2}\right)\right],$$

$$R_y = 2Q_2^r\left[\omega_z \frac{\rho}{u_r} + O\left(\frac{\ddot{\mu}}{\dot{\mu}}\omega \frac{\rho^2}{u_r^2}, \frac{\dot{\omega}}{\omega} \cdot \frac{\rho^2}{u_r^2}\right)\right], \qquad (1.12)$$

$$R_z = -2Q_2^r \left[\omega_y \frac{\rho}{u_r} + O\left(\frac{\omega^2}{\dot{\omega}} \frac{\rho^2}{u_r^2}, \frac{\omega^2}{\dot{\omega}} \cdot \frac{\rho^2}{u_r^2} \right) \right].$$

For steady flow of mas, we have

$$R_x = Q_2^r, \quad R_y = 2Q_2^r \omega_z \frac{\rho}{u_r}, \quad R_z = -2Q_2^r \omega_y \frac{\rho}{u_r}. \tag{1.13}$$

Formula (1.12) shows that the substitution of the principal vector of reaction forces $\bar{R}$ rate of consumption $\bar{Q}_2^r$ results in discording terms of the order of $O\left(Q_2^r \frac{\rho^2}{u_r}\right)$.

Ny virtue of (1.6), (1.8) and (1.10), the principal moment of reaction forces (1.2) is equivalent to

$$\begin{aligned} \bar{M} = &- \bar{I}_2^r + \dot{\omega}\rho^3 \bar{c} \times \bar{I} + 2\omega\rho^2 \bar{I} \times (\bar{\omega} \times \bar{c}) + \\ &+ \omega\rho^2 \bar{I} \times (\bar{\omega} \times \bar{I}) + O\left(\dot{\omega}\frac{\rho^3}{u_r}, \omega^2 \frac{\rho^3}{u_r}\right). \end{aligned} \tag{1.14}$$

Suppose y axis is taken along the vector $\bar{h} \times \bar{I}$. Projecting the vector equation (1.14), we get

$$\begin{aligned} M_x &= O\left(\dot{\omega}\frac{\rho^3}{u_r}, \omega^2 \frac{\rho^3}{u_r}\right), \\ M_r &= I_2^r \left(1 + \frac{\dot{\omega}}{\omega^2} \frac{\rho}{u_r}\right) + Q_2^r \frac{\rho}{u_r} (2h_x \omega_y - 2h_y \omega_x + \rho\omega_y) + \\ &\quad + O\left(\dot{\omega}\frac{\rho^3}{u_r}, \omega^2 \frac{\rho^3}{u_r}\right), \\ M_z &= Q_2^r \frac{\rho}{u_r} (2h_x \omega_z - 2h_z \omega_x + \rho\omega_z) + O\left(\dot{\omega}\frac{\rho^3}{u_r}, \omega^2 \frac{\rho^3}{u_r}\right). \end{aligned} \tag{1.15}$$

If the pole is taken on longitudinal plane of the body and the body rotates about the axis perpendicular to the plane, then

$$M_x = 0\left(\frac{\rho^3}{u_r}, \frac{\rho^3}{u_r}\right), \quad M_z = 0\left(\frac{3}{u_r}, \frac{\rho^3}{u_r}\right),$$

$$M_y = l_2^r\left(1 + \frac{\omega\rho}{u_r}\right) + Q_2^r \frac{\rho}{u_r}(2h_x + \rho)\omega + 0\left(\frac{\rho^3}{u_r}, \frac{\rho^3}{u_r}\right). \tag{1.16}$$

In case of steady flow we get:

$$M_y = l_2^r + Q_2^r \frac{\rho}{u_r}(2h_x + \rho)\omega, \quad M_x = M_z = 0. \tag{1.17}$$

Formulae (1.15) - (1.17) show that the substitution of the principal moment of reaction forces $\bar{M}$ by the moment of per second consumption (discharge), results in discarding terms of the order of $0\left(Q_2^r \frac{\rho}{u_r}\right)$.

In modern chemical engines (engines using chemicals) u_r is of the order of 2000 - 4000 meters per second. ρ may also be several meters. Therefore, $\frac{\omega\rho}{u_r}$ possesses the order, 0.001. Consequently, the replacement of the principal vector and the principal moment of reaction forces by per second discharge of the relative momentum and the relative kinetic moment is possible if the body does not turn sharply and if there is no drastic change in consumption rate.

3. Remark. 1 - When the body rotates rapidly with the working motor, Carioliso force and the moments attain a large value.

Formulae (1.9), (1.10), (1.13) and (1.15) show that the order of this value is $O(Q_2^r \sim \frac{\omega^2 \rho}{u_r})$, where ω is the angular velocity of the rotation.

Remark 2. It is necessary to take into account the carioliso forces and moments in case of the prolonged motion and particularly while investigating the stability.

Remark 3. The carioliso forces and moments and also the relative accelerations of the particles definitely influence the motion of the bodies like Mesherky's rocket boat in which case ρ is of the order of the length of the boat, and the relative velocity of baling out of the particles u_r is insignificant.

Remark 4. When the motor speeds up and slows down, we will take into account the effect of the relative accelerations of the gas particles on the principal vector and the principal moment of the reaction forces, if the order of the ratio $-\frac{\dot{\omega}\rho}{\omega^2\rho}$ is close to $\frac{u_r}{\omega\rho}$.

2. Optimal Control of the Coplanar Motion of the Body with Variable mass in the Central Field.

1. Suppose the internal motion of the particles in the body is steady, then in case of the outflow of the principal vector and the principal moment of the reaction forces (1.1) and (1.2) are written in the form

$$\mathbf{R} = -\mu \bar{\mathbf{u}}_r + \mathbf{J} \qquad (2.1)$$

$$\mathbf{M} = -\mathbf{I}_2^r + \mathbf{H}. \qquad (2.2)$$

$\mu^{\bullet}$ - per second consumption (expenditure) of the mass

u_r - average outflow velocity,

$\bar{J}$ - principal vector of the carioliso forces,

$\bar{I}_2^r$ - per second discharge of the relative kinetic moment of the gas particles,

$\bar{H}$ - principal moment of the carioliso forces principal vector of the carioliso forces (1.3) can be expressed as

$$\bar{J} = 2\dot{\mu}\bar{\omega} \times \bar{\rho}, \qquad (2.3)$$

where $\bar{\omega}$ - is the anagular velocity of the rotation of the rocket; $\bar{\rho}$ - is the radius vector of the point of body from which the fuel drops with reference (respect) to the middle point of the outlet cross-section.

To study the rotation of the rocket we consider the pole at the point of the body which coincides with the center of the mass of the given time. The principal moment of the carioliso forces (1.4) about this for the plane motion of the body may be written in the form

$$\bar{H} = \bar{\omega} \times \bar{I}^r - \dot{\mu}(\rho_c^2 - \rho_T^2)\bar{\omega}, \qquad (2.4)$$

in which $\bar{I}^r$ - is the relative kinetic moment of the particles moving in the body; ρ_c - is the distance from the pole to the middle point of the cross-section, ρ_T is the distance from the pole to the point from the fuel drops.

Let us suppose the internal motion of the particles is symmetrical about the longitudinal axis. Then in case of the plane motion of the body the vectors $\bar{\omega}$ and $\bar{I}^r$ will be the collinear. Therefore, we may omit the first term on the right side of (2.4).

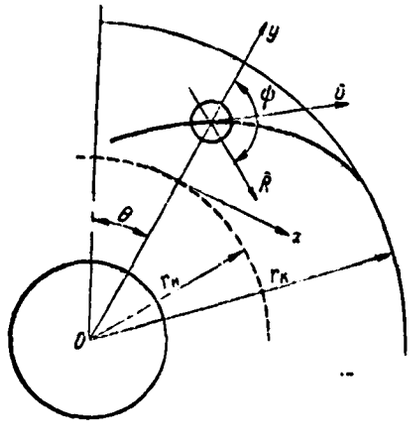

Рис. 12

Figure - 12

Let us discuss the problem of determining a short duration plane for the body of variable mass with optimum change into a circular orbit in vacuum around a spherically symmetrical body. Optimum in this case should be understood as the least expenditure of mass.

Consider the system of the polar coordinates $(r_H + y, \theta)$, where r_H is distance of the body from the center at the start of the motion. The polar axis is drawn from the center when the body is in the initial position.

Let ψ denote the angle at which the reaction force $\bar{R}$ is inclined to the radius vector of center of the mass. Angle ψ (fig. 12) is measured from O_y in the direction of the velocity of the center of mass $\bar{v}$. g denotes the acceleration of the force of the attraction gravitation on distance r_H. We know that $\vartheta = \varphi + \psi$.

Suppose m - mass, $v_x = (r_H + y)\dot{\varphi}$, $v_y = \dot{y}$. Project the equation of the motion of the inertial center of the body in section § 6 of chapter II on the generalized axes of the polar coordinates. Then by virtue of (2.1) and (2.3) we get

$$m\left(\dot{v}_y - \frac{v_x^2}{r_H+y}\right) = -mg\frac{r_H^2}{(r_H+y)2} + \dot{\mu} u_r \cos\psi - 2\dot{\mu}\left(\dot{\varphi} + \frac{v_x}{r_H+y}\right)\sin\psi,$$

$$m\left(\dot{v}_x + \frac{v_x v_y}{r_H+y}\right) = \dot{\mu} u_r \sin\psi + 2\dot{\mu}\left(\dot{\varphi} + \frac{v_x}{r_H+y}\right)\cos\psi. \qquad (2.5)$$

In short, we suppose that the reaction force acts along the longitudinal axis of the body. Turning of the body is brought about by the auxiliary motor, let $\dot{\mu}_1$ denote the per second consumption of mass, ρ_1 denote the arm of reaction force in respect of the center of mass of the body and let u_{r1} denote the effective outflow velocity of gas. Then we will have $l_2^r = \dot{\mu}_1 u_r \rho_1$.

Equation of rotation of the body about the point which coincides with the center of the mass at a given time, which was obtained in section § 6 of Chapter II is now written with the help of (2.4) as

$$I\left[\ddot{\vartheta} + \frac{\dot{v}_x}{r_H + y} - \frac{v_x v_y}{(r_H + y)^2}\right] = L + \dot{m}_1 u_{r1} l_1 - (\rho_c^2 - \rho_T^2)\left(\dot{\vartheta} + \frac{v_x}{r_H + y}\right). \tag{2.6}$$

I - moment of the inertia of the body,

L - moment of the usual forces, for example, those caused by the rotation of the special fly wheels,

Index "1" - denotes the value pertaining to booster engine.

In general, the simultaneous consideration of equations (2.5) and (2.6) minimises the consumption of the mass. It will be seen that the consumption of the mass cuased by the basic change in the direction and the magnitude of the velocity of center of the mass and the turning of the body in accordance with equation (2.6) involve little expenditure of energy. Under these conditions, if the equation (2.5) determines the optimal program for $\dot{v}$ and $\dot{\vartheta}$, then equation (2.6) enables us to select the corresponding program for L and $\dot{m}_1$.

In each case, equation (2.6) shows that less energy will be used for turning the frame of the rocket, if not only ϑ but also $\dot{\vartheta}$ are adequately small.

2. Motion with the Drastic Change of Reaction Force.

We will find how the angle of reaction force $\vartheta_p(t)$, changes if a rapid transformation (transition) of the body from the initial state r_H, v_x^o, v_y^o on the circular trajectory is brought about when the consumption of the mass is the least. Denote the time of motion

by T. With the above assumption T is small. We may think that during the motion the distance of the body from the center practically does not change and remains equal to r_H.

In case of fast motion we can neglect in equations (2.5) the difference between the force of attraction (gravitation) and the centrifugal force in comparison with the reaction force. Similarly we can also neglect other terms coming in due to non-uniformity of the gravitational field. $\nu = \log m$. Then in view of (2.5) we get the simplified equations of motion as

$$\dot{v}_y = -\nu u_r \cos\psi + 2\nu\rho\dot{\psi}\sin\psi, \qquad (2.7)$$

$$\dot{v}_x = -\nu u_r \sin\psi - 2\nu\rho\dot{\psi}\cos\psi .$$

Further we have the following conditions

$$\int_0^T (\nu u_r \cos\psi - 2\nu\rho\dot{\psi}\sin\psi)\, dt = v_y^0,$$

$$\int_0^T (\nu u_r \sin\psi + 2\nu\rho\dot{\psi}\cos\psi)\, dt = v_x^0 - \sqrt{gr_H}. \qquad (2.8)$$

In (2.8) the integrands are the explicit functions of time.

The problem posed reduces to finding the minimum of the function with constant undermined multipliers.

$$\int_0^T \left[-\nu + \lambda_1 (\nu u_r \cos\psi - 2\nu\rho\dot{\psi}\sin\psi) + \lambda_2 (\nu u_r \sin\psi + 2\nu\rho\dot{\psi}\cos\psi)\right] dt. \qquad (2.9)$$

Euler's equation for the variables $\dot{\nu}$ and ψ assumes the form:

$$u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi) - 2\omega (\lambda_1 \sin\psi - \lambda_2 \cos\psi) = 1. \qquad (2.10)$$

$$(\dot{\nu} u_r - 2\omega\dot{\nu}) (\lambda_1 \sin\psi - \lambda_2 \cos\psi) = 0. \qquad (2.11)$$

Equation (2.11) is satisfied if: (I) $\dot{\nu} = 0$, (II) $\lambda_1 \sin\psi - \lambda_2 \cos\psi = 0$ and (III) $\dot{\nu} u_r - 2\omega\dot{\nu} = 0$.

When (I) holds, it means free motion, and this case is not of interest. Because t is in seconds $p \leq t_p$, then (II) shows an unsteady flow of particles and must be given particular attention. When the outflow of the particles is unsteady formula (2.1) and (2.2) for the principal vector and the principal moment of reaction forces prove to be inadequate. Therefore, we confine ourselves to the consideration of case (II).

Equations (2.10) and (2.11) are replaced by:

$$\begin{aligned} &\lambda_1 \sin\psi - \lambda_2 \cos\psi = 0, \\ &u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi) = 1. \end{aligned} \qquad (2.12)$$

These equations show that $\psi = \psi_0 = \text{constant}$. Conditions (2.8) are then equivalent to :

$$\begin{aligned} u_r (\nu_T - \nu_0) \cos\psi_0 &= v_y^0, \\ u_r (\nu_T - \nu_0) \sin\psi_0 &= v_x^0 - \sqrt{gr_H}. \end{aligned} \qquad (2.13)$$

The yield

$$m_T = m_0 e^{-\frac{V}{u_r}}, \quad V = \sqrt{v_y^{o2} + (\sqrt{gr_H} - v_x^o)^2}. \qquad (2.14)$$

Formulae (2.13) and (2.14) provide the fast optimal device. In view of the supposed steady outflow we may make the rule of the consumption similar to that of the uniform burning in = constant. The motion may take place for any length of time; shorter the time, closer will be the real conditions of motion to those imagined.

3. Motion with slight change of reaction force. Suppose the motor is such that cannot develop acceleration greater than g'. We will find the optimal plane transformation of the body on the circular orbit of radius r_K. In such a case the conditions of motion of the body and as in (1) we may drop the terms, inequations (2.6), contributed by the carioliso forces of the internal motion of the particles. In view of a slight change in reaction force, we may say, on the basis of conclusions in section §1, that the forces produced by unsteady motion of particles are smaller as composed with the per second discharge of the relative momentum m = const.

Equations (2.5) assume the form

$$\dot{v}_y = \frac{v_x^2}{r_H + y} - \frac{gr_H^2}{(r_H + y)^2} - \varsigma u_r \cos\varphi,$$

$$\dot{v}_x = -\frac{v_x v_y}{r_H + y} - \varsigma u_r \sin\varphi, \quad \dot{y} = v_y. \qquad (2.15)$$

Interpreting optimality as the least. Consumption of mass, we will minimize the functional

$$-\int_0^T \dot{\nu}\, dt = \ln \frac{m_o}{m_T}.$$

T - denotes the time during which the motion takes place and this may or may not be known.

The variational problem is dealt with in terms of the maximum principale [75]. Poutryagin's function of our problem.

$$H = \dot{\nu}\,(1 - \lambda_1 u_r \cos\psi - \lambda_2 u_r \sin\psi) + \\ + \lambda_1 \left[\frac{v_x^2}{r_H + y} - \frac{gr_H^2}{(r_H + y)^2}\right] - \lambda_2 \frac{v_x v_y}{r_H + y} + \lambda_3 v_y \qquad (2.16)$$

must be treated as Hamiltons function with impulses λ_i and coordinates v_y, v_x, y.

Corresponding canonical equations yield equations of motion (2.15) and Euler's equations

$$\dot{\lambda}_1 = \frac{v_x}{r_H + y}\lambda_2 - \lambda_3,$$

$$\dot{\lambda}_2 = -\frac{2v_x}{r_H + y}\lambda_1 + \frac{v_y}{r_H + y}\lambda_2, \qquad (2.17)$$

$$\dot{\lambda}_3 = -\left[\frac{2gr_H^2}{(r_H + y)^3} - \frac{v_x^2}{(r_H + y)^2}\right]\lambda_1 - \frac{v_x v_y}{(r_H + y)^2}\lambda_2.$$

According to the maximum principle theorem, the function H with variables ν and ψ has a point of the maximum on optimal trajectory.

To obtain interior extremum, if such a point exists, we have

$$\frac{\partial H}{\partial \dot{v}} = 1 - u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi) = 0,$$

$$\frac{\partial H}{\partial \psi} = \dot{v}\, u_r (\lambda_1 \sin\psi - \lambda_2 \cos\psi) = 0.$$

Calculate the second derivatives of function H

$$\frac{\partial^2 H}{\partial \dot{v}^2} = 0, \quad \frac{\partial^2 H}{\partial v \partial \psi} = u_r (\lambda_1 \sin\psi - \lambda_2 \cos\psi),$$

$$\frac{\partial^2 H}{\partial \psi^2} = \dot{v}\, u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi).$$

The necessary and sufficient conditions for the function H to be maximum show that the extremal may consist of the parts of the following trajectoris:

Free motion if

$$\dot{v} = 0, \; 1 - u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi) \geqslant 0, \; \lambda_1 \sin\psi - \lambda_2 \cos\psi = 0; \quad (A)$$

Motion with variable reaction acceleration (in value or magnitude)

$$\lambda_1 = \frac{1}{u_r} \cos\psi, \quad \lambda_2 = \frac{1}{u_r} \sin\psi; \quad (B)$$

Motion with largest reaction acceleration g, when the following conditions are satisfied

$$\dot{v} = -\frac{g'}{u_r}, \; 1 - u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi) \leqslant 0, \; \lambda_1 \sin\psi - \lambda_2 \cos\psi = 0. \quad (c)$$

In order to carry out optimal motion of the body on circular trajectory with the help of trajectory with changing acceleration, formula (B) must satisfy the canonical system (2.15) and (2.17) when $t = 0$, $v_y = v_y^o$, $v_x = v_x^o$, $y_o = 0$; $t = T$, $v_y^T = r_H\sqrt{\frac{g}{r_K}}$, $y_T = r_K - r_H$. Since H does not depend on t, therefore, from the canonical system it follows that integral H = constant which is satisfied by (B).

Differentiating the first equation of the system (2.17) and substituting the value of λ_3, the system (2.17) is reduced to two equations in terms of λ_1 and λ_2. Putting the values of λ_1 and λ_2 from (B), the system (2.17) is transformed into

$$\dot{\psi}\,\mathrm{ctg}\,\psi = \frac{v_y}{r_H + y} - \frac{2v_x}{r_H + y}\,\mathrm{ctg}\,\psi,$$

$$\ddot{\psi} = -\dot{\psi}^2\,\mathrm{ctg}\,\psi + \frac{u_r}{r_H + y}\sin\psi + \left[\frac{3v_x^2}{(r_H + y)^2} - \frac{2gr_H^2}{(r_H + y)^3}\right]\mathrm{ctg}\,\psi. \tag{2.18}$$

Differentiate first equation of the system (2.18) and putting the value of $\dot{\psi}$ from second equation, we get an equation of second degree in $\dot{\psi}$. Assuming that $\psi \neq \frac{\pi}{2}$, We get

$$\frac{1}{(r_H + y)^2}\left[(v_y - v_x\,\mathrm{ctg}\,\psi)^2 - \frac{gr_H^2}{r_H + y}(1 - 2\,\mathrm{ctg}^2\psi)\cos^2\psi\right] = 0. \tag{2.19}$$

The relation (2.19) is obtained as result of comparing the terms of order r_H^{-2} in the too equations in obtained in different ways. Therefore, formula (2.19) is valid only if the terms upto order r_H^{-2} inclusive are repressed in the equations.

On trajectories with program reaction acceleration, in view of (2.16) and (2.17), we have

$$H = \frac{1}{u_r}\left\{\cos\psi\left[\frac{v_x^2}{r_H + y} - \frac{gr_H^2}{(r_H + y)^2}\right] + \psi v_y \sin\psi\right\} = h = \text{const.} \tag{2.20}$$

The principle of maximum states that for indefinite manoeuvre time T we have h = 0, and when the time T is known h may be different from zero. However, in this case also the boundary conditions in conjunction with (2.20) make h = 0.

Using the first equation of (2.18), formula (2.20) is transformed into

$$H = \frac{\sin^2}{u_r (r_H + y)\cos\psi}\left[(v_y - v_x \operatorname{ctg}\psi)^2 - \frac{gr_H^2}{r_H + y}\operatorname{ctg}^2\psi\right] = 0. \tag{2.21}$$

Formulae (2.19) and (2.21) can not be satisfied unless $\psi \neq \frac{\pi}{2}$. When $\psi = \frac{\pi}{2}$ equation (2.18) gives $v_y = 0$. Since $v_y = y$, we have y = constant. Consequently, the body must move in a circle. However, the reaction force on a coplanar circular trajectory in central field must be equal to zero. We can not have directed motion. Therefore, the exact optimal coplanar displacement of the body on circular trajectory can not be a trajectory with programed acceleration. Consequently, the exact optimal manoevure of the motion consits of the trajectory of free motion (A) and the trajectory of largest reaction acceleration (C). Application of non-coplanar transition trajectories, all points of which are equally distant from the

attracting center, has been discussed by A.I. Lurye in his works [42].

Formula (2.19) is not employed to find approximate optimal motions for which the exactness of the function $\psi(t)$ is less than $(\frac{r_K - r_H}{r_H})^2$. In paragraphs 4 and 5 approximate conditions are worked out for the case (B), which satisfy the equations (2.15), (2.17) and the relation (2.20) when $h = 0$. These conditions may include, in particular, largest reaction acceleration and largest reaction force.

For the treatment of approximate optimal conditions we set up

$$\xi = r_H \sqrt{\frac{\varepsilon}{r_H + y}} - v_x.$$

Then the system (2.15) is written in a more convenient form

$$\dot{v}_y = -\frac{2r_H\sqrt{\varepsilon}\,\xi}{(r_H + y)^{\frac{3}{2}}} + \frac{\xi^2}{r_H + y} - \dot{v}\,u_r\cos\psi,$$

$$\dot{\xi} = \tfrac{1}{2}\frac{r_H\sqrt{\varepsilon}\,v_y}{(r_H + y)^{\frac{3}{2}}} - \frac{\xi v_y}{r_H + y} + v\,u_r\sin\psi, \qquad (2.22)$$

$$\dot{y} = v_y.$$

v_y, ξ, and y have the same infinitesimal order.

4.

so that the terms of the order $\left(\frac{r_K - r_H}{r_H}\right)^{\frac{1}{2}}$ do not attain considerably high magnitude in the manoeuvre time T.

Express v_x in terms of ξ and then in accordance with (2.18), $\psi = \psi_o + O\left(r_H^{-\frac{1}{2}}\right)$. Equations (2,22) are then written in the form

$$\dot{v}_y = \nu u_r \cos\psi_o + O\left(r_H^{-\frac{1}{2}}\right).$$

$$\dot{\xi} = \dot{\nu} u_r \sin\psi_o + O\left(r_H^{-\frac{1}{2}}\right).$$

Therefore, precisely upto the terms of order $\left(\frac{r_K - r_H}{r_H}\right)^{\frac{1}{2}}$. We arrive at the results (2.13) and (2.14) valid for abrupt manoeuvre. Satisfying the boundary condition on the variable y, we get the condition

$$v_y^o T - u_r \cos\psi_o \int_o^T (\nu - \nu_o)\, dt = r_K - r_H + O\left(r_H^{-\frac{1}{2}}\right). \tag{2,23}$$

Formula (2.23) shows that if v_y^o and $r_K - r_H$ do not have order $r_H^{-\frac{1}{2}}$, then $\cos\psi_o \neq 0$ and $\psi_o \neq \pm\frac{\pi}{2}$. The approximate optimal conditions thus obtained satisfy (2.20) when h = 0. Actually, the function H in the result (2.20) turns out to be $H = 0 + O\left(r_H^{-\frac{1}{2}} u_r^{-1}\right)$.

If the manoeuvre time T is given, the relations (2.14) and (2.23) must be satisfied by the law of change of mass depending on one parameter and then we choose this parameter and the time T in

such a way that the law satisfies the conditions (2.14) and (2.23).

Example 1. Let us take the law of change of mass $m = m_0 e^{-\alpha t}$. Then conditions (2.14) and (2.23) assume the form

$$\alpha u_r T = V, \quad \alpha u_r T^2 \cos\psi_0 + 2v_y^0 T - 2(r_K - r_H) = 0.$$

This gives

$$T = \frac{2(r_K - r_H)}{3v_y^0}, \quad \alpha = \frac{V}{u_r T}.$$

Example 2. Consider the direction including the trajectory of largest reaction acceleration g'. In the interval $(0, \tau)$ we have $v = v_0$; and in conformity with this for the interval $[\tau, T]$ we write $v = v_0 - \frac{g'}{u_r}(t - \tau)$.

Conditions (2.14) and (2.23) enable us to determine T and in the form

$$T = \frac{r_K - r_H}{v_y^0} + \frac{V}{2g'}, \quad \tau = \frac{r_K - r_H}{v_y^0} - \frac{V}{2g'}.$$

Example 3. Take the direction including the trajectory of largest reaction force. In the interval $(0, \tau)$ $v = v_0$ and accordingly in the interval (τ, T) $v = 1n\left\{m_0\left[1 - \beta(t - \tau)\right]\right\}$. T and τ are equal to

$$T = \frac{r_K - r_H}{v_y^0} + \frac{u_r}{\beta V}\left[1 - \left(1 + \frac{V}{u_r}\right) e^{-\frac{V}{u_r}}\right], \tau = T - \frac{1}{\beta}\left(1 - e^{-\frac{V}{u_r}}\right).$$

If the values $\frac{v_y^o}{\sqrt{gr_H}}$, $\frac{\xi_o}{\sqrt{gr_H}}$, $\left(\frac{r_K - r_H}{r_H}\right)^{\frac{1}{2}}$ are relatively high then we may get approximate optimal manoeuvre with increased exactness.

Equations (2.22) are written in the form

$$\begin{aligned} \dot{v}_y &= -2\Omega\xi - \nu u_r \cos\psi + 0\,(r_H^{-1}), \\ \dot{\xi} &= \frac{1}{2}\Omega v_y + \nu u_r \sin\psi + 0\,(r_H^{-1}). \end{aligned} \tag{2.24}$$

Here $\Omega = \sqrt{g}\, r_H r_K^{-\frac{3}{2}}$ is angular velocity of the body on finite circular trajectory. This value is of the order of $r_H^{-\frac{1}{2}}$.

Equations (2.18) become

$$\begin{aligned} \dot{\psi}\,\mathrm{ctg}\psi &= -2\Omega\,\mathrm{ctg}\psi + 0\,(r_H^{-1}), \\ \dot{\psi} &= -\psi^2\,\mathrm{ctg}\psi + 0\,(r_H^{-1}). \end{aligned} \tag{2.25}$$

We consider the case when the value of ψ if not close to $\frac{\pi}{2}$. Equations (2.25) have a unique solution

$$\psi = \psi_o - 2\Omega t + 0\,(r_H^{-1}). \tag{2.26}$$

We will find the solution of equations (2.24) in series Ω and restrict it to the first two terms.

$$v_y = v_y^o - u_r(\nu - \nu_o)\cos\psi_o - 2\Omega\left[\xi_o + u_r(\nu - \nu_o)\sin\psi_o\right]t + 0\,(r_H^{-1}),$$

$$\xi = \xi^o + u_r(\nu - \nu_o)\sin\ _o + \frac{1}{2}\ v_y^o t + \frac{3}{2}\ u_r \cos\ _o \int_o^t (\ - \ _o)\,dt -$$

$$- 2\Omega u_r (\nu - \nu_o)\, t \cos\psi_o + O(r_H^{-1}),$$

$$y = v_y^o t - \Omega \xi_o t^2 - u_r \cos\psi_o \int_o^t (\nu - \nu_o)\, dt -$$

$$- 2\Omega u_r \sin\psi_o \int_o^t t\,(\nu - \nu_o)\, dt + O(r_H^{-1}). \qquad (2.27)$$

The boundary conditions: $v_y(T) = \xi(T) = 0$, $y(T) = r_K - r_H$ are satisfied.

The last equation of the system (2.27) yields

$$\int_o^T (\nu - \nu_o)\, dt = - \frac{r_K - r_H - v_y^o T}{u_r \cos\psi_o} + O\left(r_H^{-\frac{1}{2}}\right). \qquad (2.28)$$

Therefore, in view of the first two equations of the system (2.27) we find

$$v_y^o - 2\Omega \xi_o T = u_r (\nu_T - \nu_o)(\cos\psi_o + 2\Omega T \sin\psi_o) + O(r_H^{-1}),$$

$$\xi_o + 2\Omega v_y^o T - \frac{3}{2}\Omega (r_K - r_H) = u_r (\nu_T - \nu_o)$$

$$(-\sin\psi_o + 2\Omega T \cos\psi_o) + O(r_H^{-1}). \qquad (2.29)$$

Equations (2.26) - (2.29) enable us to find an expression for the mass at the end of the motion

$$m_T = m_o \exp\left\{ - \frac{V}{u_r} \left[1 - \frac{3\Omega \xi_o}{2V2} (r_K - r_H) \right] \right\} + O(r_H^{-1}). \qquad (2.30)$$

Set $\psi_o = \psi'_o + \psi''_o$, in (2.29) where ψ'_o is obtained from (2.13). We then get the initial angle of inclination of reaction force

$$\psi''_o = \frac{3\Omega v_y^o}{2V^2}(r_K - r_H).$$

The last equation of the system (2.27) in conjunction with (2.28) gives the condition

$$u_r \cos\psi_o \int_o^T (\nu - \nu_o)\,dt - 2\Omega u_r \sin\psi_o \int_o^T dt_1 \int_o^{t_1} (\nu - \nu_o)\,dt -$$

$$- v_y^o T (1 - 2\Omega T \operatorname{tg}\psi_o) + \Omega \xi_o T^2 +$$

$$+ (r_K - r_H)(1 - 2\Omega T \operatorname{tg}\psi_o) = 0 + 0(r_H^{-1}). \qquad (2.31)$$

Conditions (2.30) and (2.31) can be satisfied by a set T of the law of change of mass as are the conditions (2.14) and (2.23).

The function H used in (2.20) turns out to be equal to (in view of (2.24) and (2.25)

$$H = -\frac{2\Omega}{u_r}(\xi_o \cos\psi'_o + v_y^o \sin\psi'_o) + 0(r_H^{-1}u_r^{-1}).$$

Angle ψ'_o is determined from (2.13) and, therefore, we get

$$H = 0 + 0(r_H^{-1}u_r^{-1}).$$

5.

Consider the optimal direction when ψ is close to $\frac{\pi}{2}$. Put $\varepsilon = \psi - \frac{\pi}{2}$ and assume that ε is is of order (see para 4). Denote by O_n the values which are of the order of Ω^n.

Equations of motion (2.22) can be put in the form

$$\dot{v}_y = -2\Omega\xi + \frac{\xi^2}{r_K} + \dot{\nu} u_r \varepsilon + O_3,$$
$$\dot{\xi} = \frac{1}{2}\Omega v_y - \xi\frac{v_y}{r_K} + \dot{\nu} u_r \left(1 - \frac{\varepsilon^2}{2}\right) + O_3. \qquad (2.32)$$

(2.18) can be written in the form

$$\varepsilon\dot{\varepsilon} = -\frac{v_y}{r_K} - 2\varepsilon\Omega + \frac{2\varepsilon\xi}{r_K} + O_4, \quad \ddot{\varepsilon} = \dot{\nu}\frac{u_r}{r_K} + O_3. \qquad (2.33)$$

Differentiate first relation in (2.33) and then using equations (3.32) and (2.33) we obtain a second degree equation in which when solved will give

$$\dot{\varepsilon} = -2\Omega + \frac{\xi}{r_K} + O_3. \qquad (2.34)$$

Formula (2.34) and the second equation of the system (2.32) enable us to determine the law of change of inclination angle of the reaction force

$$\varepsilon = \varepsilon_o - 2\Omega t + \frac{\xi_o}{r_K} t + \frac{u_r}{r_K}\int_o^t (\nu - \nu_o)\, dt + O_3,$$
$$O(\varepsilon_o) = O_1. \qquad (2.35)$$

Integrating equation (2.32), we get

$$\xi = \xi_0 + u_r(\nu - \nu_0) + \frac{1}{2}\int_0^t (\varepsilon_0 - 2\Omega t)\{\Omega[\xi_0 + u_r(\nu - \nu_0)] - (\varepsilon_0 - 2\Omega t)\nu u_r\}dt + O_3, \quad v_y = \varepsilon\xi + \zeta_0 + O_3,$$

$$\zeta_0 = v_y^0 - \varepsilon^0\xi^0 = O_2, \quad y = \zeta_0 t + \int_0^t \left[\varepsilon_0 - 2\Omega t_1 + \frac{\xi_0 t_1}{r_K} + \frac{u_r}{r_K}\int_0^{t_1}(\nu - \nu_0)\,dt_2\right][\xi_0 + u_r(\nu - \nu_0)]dt_1 + O_3. \tag{2.36}$$

Formulae (2.36) show that the optimal direction takes place only when v_y^0 and $r_K - r_H$ are infinitesimals of first order.

Satisfying boundary and initial conditions, we get

$$u_r \ln\frac{m_0}{m_T} - \frac{1}{2}\int_0^T (\varepsilon_0 - 2\Omega t)\{\Omega[\xi_0 + u_r(\nu - \nu_0)] - (\varepsilon_0 - 2\Omega t)\nu u_r\}dt = \xi_0 + O_3,$$

$$\zeta_0 = O_3, \quad \int_0^T \left[\varepsilon_0 - 2\Omega t + \frac{\xi_0 t}{r_K} + \frac{u_r}{r_K}\int_0^t(\nu - \nu_0)\,dt_1\right] \times [\xi_0 + u_r(\nu - \nu_0)]dt = r_K - r_H + O_3, \tag{2.37}$$

$$\varepsilon_0 = \frac{v_y^0}{\xi_0} + O_3, \quad \xi_0 = \sqrt{gr_H - v_x^0}. \tag{2.38}$$

If the last equations are not known, then the relations (2.37) enable us to select the parameters of the law of change of mass and the time of motion T.

Let us check the conditions (2.20). Using (2.35) to express angle ψ in (2.20) we get

$$H = \frac{2\Omega}{u_r}(\varepsilon\xi - v_y) + u_r^{-1} O_3.$$

In accordance with (2.36) we find that

$$H = 0 + u_r^{-1} O_3.$$

If we restrict to terms of first order of infinitesimal then (2.37) will be written as

$$m_T = m_0 e^{-\frac{\xi_0}{u_r}} + O(r_H^{-1}),\ \varepsilon \neq \varepsilon_0 - 2\Omega t + O(r_H^{-1}),$$

$$u_r(\varepsilon_0 - 2\Omega T)\int_0^T (\nu - \nu_0)\,dt + 2\Omega u_r \int_0^T dt \int_0^t (\nu - \nu_0)\,dt_1 +$$

$$+ \xi_0 T(\varepsilon_0 - \Omega T) = r_K - r_H + O(r_H^{-1}). \qquad (2.39)$$

Example 4. Let us see how the law of change of mass $m = m_0 e^{-\alpha t}$ is applied when ψ is close to $\frac{\pi}{2}$ and when direction has preciseness of order r_H^{-1}.

In accordance with (2.38) and (2.39) we get

$$\alpha u_r T = \xi_0,\ \xi_0 = \sqrt{\varepsilon r_H} - v_x^0,\ \varepsilon = \varepsilon_0 - 2\Omega t,\ \varepsilon_0 = \frac{v_y^0}{\xi_0},$$

$$-\frac{2}{3}\alpha\Omega u_r T^3 - \frac{1}{2}(2\Omega\xi_0 + \alpha\varepsilon_0 u_r)T^2 + \varepsilon_0\xi_0 T = r_K - r_H.$$

Eliminating from second equation , we get a second degree equation in T which gives two values of T

$$T_{1,2} = \frac{3}{4}\,\frac{\varepsilon_0}{\Omega}\left(1 \pm \sqrt{1 - \frac{16\Omega(r_K - r_H)}{3\varepsilon_0^2 \xi_0}}\right).$$

If $v_y^0 = 0$, $r_K - r_H = 0$, and $\xi_0 \neq 0$, then we can not have optimal direction. If $r_K - r_H = 0$, and $v_y^0 \neq 0$, then $T = \frac{3}{2}\,\frac{\varepsilon_0}{\Omega}$. In such an event $\xi_0 \neq 0$, $= \frac{2\Omega\xi_0}{3\varepsilon_0 u_r}$.

Example 5. Suppose the conditions are the same as in example 4, then we will examine the direction including the trajectory of largest reaction acceleration. To determine the time of free motion T and the time of entire motion T, we have, in accordance with (2.38) and (2.39)

$$\tau = T - \frac{\xi_0}{g'}, \quad T^2 - \left(\frac{\varepsilon_0}{\Omega} + \frac{\xi_0}{g'}\right) T + \frac{\varepsilon_0 \xi_0}{2\Omega g'} + \frac{\xi_0^2}{3g'^2} + \frac{r_K - r_H}{\Omega \xi_0} = 0.$$

Similarly, we can deal with the direction including the trajectory of largest reaction force. (see example 3).

3. Optimal Direction of Non-Coplanar Motion of a Body with Variable Mass in the Central Field.

1. Equations of motion. Let us determine the optimal displacement of the body with variable mass on a given circular trajectory in the central field when the decrease (3.1) of mass is minimum.

The position of the body is fixed by spherical coordinates $(r_H + y, \varphi, \theta)$, where r_H is the distance of the body from the attracting center at the beginning of the motion. Denote by r_K the radius of the given circular trajectory and take the plane of trajectory as the plane of reading. (fig.13). For convenance, suppose the reaction force $\bar{N}$ acts along an arbitrary axis of the body. The rotation in the body is brought about by auxiliary motor of small power or by special fly wheels.

The direction in which the reaction force acts is determined by two angles: ψ the angle between $\bar{N}$ and the radius-vector and γ the angle between a line parallel to $\bar{N}$ and the tangent to the great circle whose plane passes through $\bar{N}$.

We will assume that the internal motion of particles in the body has been set up. Projecting the equation of motion of inertial center of the body (given in sactions, chapter II) on generalized axes of spherical system of conditions, we get

$$m\left[\dot{v}_y - (v_\varphi^2 + v_\theta^2)(r_H + y)^{-1}\right] = -mgr_H^2(r_H + y)^{-2} + \\ + \mu u_r \cos\psi + J_r,$$

$$m\left[\dot{v}_\varphi + v_\varphi(v_y - v_\theta \operatorname{tg}\theta)(r_H + y)^{-1}\right] = \mu u_r \sin\psi\cos\gamma + J_\varphi,$$

$$m\left[\dot{v}_\theta + (v_\theta v_y + v_\varphi^2 \operatorname{tg}\theta)(r_H + y)^{-1}\right] = \mu u_r \sin\psi\sin\gamma + J_\theta. \quad (3.1)$$

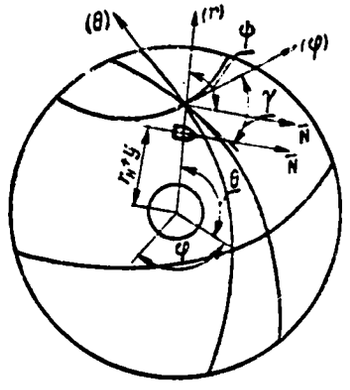

Рис. 13

Figure - 13

In equations (3.1), we have m - mass of the body, $v_y = \dot{y}$, $v_\varphi = (r_H + y)\dot{\varphi}\cos\vartheta$, $v_\vartheta = (r_H + y)\dot{\vartheta}$, g - acceleration of attractive force at height r_H, μ - per second expenditure of mass, u_r - constant effective velocity of gas discharge. J_r, J_φ and J_ϑ denote the projections of principal vector of carioliso forces brought into play by the motion of particles in the body.

In accordance with (1.3), principal vector of carioliso forces is written as

$$\mathbf{J} = 2\mu\,(\bar{v}_T^e - \bar{v}_c^e).$$

where $\bar{v}_T^e$ is the velocity of that point of the body from which the fuel falls. $\bar{v}_c^e$ - is the velocity of the middle point of the outlet cross section.

In accordance with Euler's formula for velocity of points, we have

$$\bar{J} = 2\mu\rho\ (\bar{\omega} \times \bar{i}).$$

ρ - is the length of the space inside the body occupied by moving gases; $\bar{\omega}$ - is angular velocity of rotation of the body $\bar{i}$ - is unit vector along any axis of the body, which is in the same direction as the rotation force.

Projections of $\bar{\omega}$ are given by

$$\omega_r = \dot{\gamma} + \frac{v_\varphi \operatorname{tg}\theta}{r_H + y},$$

$$\omega_\vartheta = -\dot{\psi}\sin\gamma - \frac{v_\vartheta}{r_H + y},$$

$$\omega_\theta = \dot{\psi}\cos\gamma + \frac{v_\varphi}{r_H + y}.$$

These enable us to find the projections of principal vector of Cariolis forces. They are

$$J_r = -2\mu\rho\sin\psi\left[\dot{\psi} + (v_\varphi\cos\gamma + v_\vartheta\sin\gamma)(r_H + y)^{-1}\right],$$

$$J_\varphi = 2\mu\rho\left[\dot{\psi}\cos\psi\cos\gamma - \dot{\gamma}\sin\psi\sin\gamma + \right.$$

$$\left. + v_\varphi(\cos\psi - \operatorname{tg}\theta\sin\psi\sin\gamma)(r_H + y)^{-1}\right],$$

$$J_\vartheta = 2\mu\rho\left[\dot{\psi}\cos\psi\sin\gamma + \dot{\gamma}\sin\psi\cos\gamma + \right.$$

$$\left. + (v_\varphi \operatorname{tg}\theta\sin\psi\cos\gamma + v_\vartheta\cos\psi)(r_H + y)^{-1}\right]. \quad (3.2)$$

In order to study the rotation of the body Euler's equations may be deduced from section § 6 of chapter II. Suppose the internal motion of particles is symmetrical about an arbitrary axis and center of mass is located on this axis. Then the relative kinetic moment of particles $\bar{I}^r$ calculated about the center of mass is zero. The principal moment of carioliso forces, in accordance with (1.4) is given by

$$\bar{R} = -\mu \left(\rho_c^2 - \rho_T^2 \right) \bar{\omega}_1. \qquad (3.3)$$

ρ_c - distance from the center of mass to the output cross section; ρ_T - distance between the center of mass and the point from which fuel falls; $\bar{\omega}_1$ - Vectorial component of angular velocity of rotation of the body perpendicular to its axis.

Formula (3.3) shows that moment of carioliso forces of internal motion of particles is equivalent to the moment due to the resistance of liquid medium. Therefore, for least expenditure of mass we should give a V-turn to the body either with disconnected motor or with very small angular velocity.

As in section 2, we will think that the loss of mass is fundamentally brought about by the change in magnitude and direction of velocity of the center of mass of the body and that for a U-turn the body requires little amount of energy.

2. Motion Under Sudden Change in Reaction Force.

We will find the function of change for the angles of inclinations of reaction force ψ (t) and γ (t) which effects a rapid transition of the body from the initial stage v_y°, v_φ°, v_ϑ°, r_H, θ° to the circular trajectory of redius r_K with the least loss of mass. Denote the time of motion by T. Since T is small, distance of the body from the attracting center does not change $r_K = r_H$ and the angle θ is not changed.

To determine rapid optimal maneuve we neglect in equations (3.1) and formula (3.2) the difference of attraction and centrifugal forces and also those terms which come in due to non-homogeneous gravitational field as compared to the reaction force. $\nu = \log m$, and we get the following equation of motion of the body.

$$\begin{aligned}
\dot{v}_y &= -\nu^* u_r \cos\psi + 2\dot{\nu}\rho\,\dot{\psi}\sin\psi, \\
\dot{v}_\varphi &= -\dot{\nu} u_r \sin\psi\cos\gamma - 2\dot{\nu}\rho\,(\dot{\psi}\cos\psi\cos\gamma - \dot{\gamma}\sin\psi\sin\gamma), \qquad (3.4) \\
\dot{v}_\vartheta &= -\dot{\nu}\, u_r \sin\psi\sin\gamma - 2\dot{\nu}\rho\,(\dot{\psi}\cos\psi\sin\gamma + \dot{\gamma}\sin\psi\cos\gamma).
\end{aligned}$$

The conditions are

$$\begin{aligned}
&\int_0^T (\dot{\nu} u_r \cos\psi - 2\dot{\nu}\rho\,\dot{\psi}\sin\psi)\,dt = v_y^\circ, \\
&\int_0^T \left[\dot{\nu} u_r \sin\psi\cos\gamma + 2\dot{\nu}\rho\,(\dot{\psi}\cos\psi\cos\gamma - \dot{\gamma}\sin\psi\sin\gamma)\right] dt = \\
&\qquad = v_\varphi^\circ - \sqrt{gr_H}, \\
&\int_0^T \left[\dot{\nu} u_r \sin\psi\sin\gamma + 2\dot{\nu}\rho\,(\dot{\psi}\cos\psi\sin\gamma + \dot{\gamma}\sin\psi\cos\gamma)\right] dt = v_\vartheta^\circ .
\end{aligned} \qquad (3.5)$$

The problem now reduces to finding the minimum of the function with constant undetermined multiplies.

$$W = \int_0^T \Big\{ -\dot{v} + \lambda_1 (\dot{v} u_r \cos\psi - 2\dot{v}\rho\dot{\psi}\sin\psi) +$$

$$+ \lambda_2 \left[\dot{v} u_r \sin\psi\cos\gamma + 2\dot{v}\rho\,(\dot{\psi}\cos\psi\cos\gamma - \dot{\gamma}\sin\psi\sin\gamma) \right] +$$

$$+ \lambda_3 \left[\dot{v} u_r \sin\psi\sin\gamma + 2\dot{v}\rho\,(\dot{\psi}\cos\psi\sin\gamma + \dot{\gamma}\sin\psi\cos\gamma) \right] \Big\} dt. \qquad (3.6)$$

Euler's equation in variable $\dot{v}$ is written as

$$u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi\cos\gamma + \lambda_3 \sin\psi\sin\gamma) +$$

$$+ 2\rho\dot{\psi}\,(-\lambda_1 \sin\psi + \lambda_2 \cos\psi\cos\gamma + \lambda_3 \cos\psi\sin\gamma) -$$

$$- 2\rho\dot{\gamma}\,(\lambda_2 \sin\gamma - \lambda_3 \cos\gamma)\sin\psi = 1. \qquad (3.7)$$

In variables of ψ and γ we obtain the following Euler's equations:

$$(\dot{v} u_r - 2\rho\ddot{v})\,(-\lambda_1 \sin\psi + \lambda_2 \cos\psi\cos\gamma + \lambda_3 \cos\psi\sin\gamma) = 0, \qquad (3.8)$$

$$(\dot{v} u_r - 2\rho\dot{v})\,(\lambda_2 \sin\gamma - \lambda_3 \cos\gamma)\sin\psi = 0. \qquad (3.9)$$

Note that the equations (3.8) and (3.9) hold for free motion when $v = 0$. Since we are considering rapid maneuver of change of velocity of the body at the given height, therefore, the maneuver including the section of free trajectory in the central field is not of much interest.

When the combustion is steady we have in m = const and $v = -\dot{v}^2$. This makes $\nu u_r - 2\rho\dot{\nu} \neq 0$ and equations (3.8) and (3.9) are equivalent to

$$-\lambda_1 \sin\psi + \lambda_2 \cos\psi\cos\gamma + \lambda_3 \cos\psi\sin\gamma = 0, \qquad (3.10)$$

$$(\lambda_2 \sin\gamma - \lambda_3 \cos\gamma)\sin\psi = 0. \qquad (3.11)$$

Equation (3.7) in accordance with the formulae (3.10) and (3.11) assumes the form

$$u_r(\lambda_1 \cos\psi + \lambda_2 \sin\psi\cos\gamma + \lambda_3 \sin\psi\sin\gamma) = 1. \qquad (3.12)$$

Equations (3.10) - (3.12) show that the c nstant angles of inclination of reaction force are responsible for rapid optimal conditions.

$$\psi = \psi_0 = \text{const}, \quad \gamma = \gamma_0 = \text{const}.$$

We satisfy the conditions (3.5)

$$(\nu_T - \nu_0)\, u_r \cos\psi_0 = v_y^0,$$

$$(\nu_T - \nu_0)\, u_r \sin\psi_0 \cos\gamma_0 = v_\varphi^0 - \sqrt{gr_H}, \qquad (3.13)$$

$$(\nu_T - \nu_0)\, u_r \sin\psi_0 \sin\gamma_0 = v_\vartheta^0 .$$

Squaring the relations (3.13) and adding, we have

$$m_t = m_0 \exp(-Vu_r^{-1}), \quad V = \sqrt{v_y^{o2} + (\sqrt{gr_H} - v_{c\varphi}^o)^2 + v_\vartheta^{o2}}. \tag{3.14}$$

In accordance with (3.13) and (3.14) angles of inclination of reaction force are given by the equations

$$\cos\psi_0 = -v_y^o V^{-1}, \quad \sin\psi_0 = V^{-1}\sqrt{(\sqrt{gr_H} - v_{c\varphi}^o)^2 + v_\vartheta^{o2}},$$

$$\cos\gamma_0 = \frac{\sqrt{gr_H} - v_{c\varphi}^o}{\sqrt{(\sqrt{gr_H} - v_{c\varphi}^o)^2 + v_\vartheta^{o2}}}, \tag{3.15}$$

$$\sin\gamma_0 = -\frac{\qquad\qquad}{\sqrt{(\sqrt{gr_H} - v_{c\varphi}^o)^2 + v_\vartheta^{o2}}}.$$

Formulae (3.14) and (3.15) determine rapid optimal maneuver. The time of motion T may be arbitrary. But the lesser it is, the more real will be the conditions of motion.

3. Motion with Gradual Change in Reaction Force.

Suppose the motor does not produce an acceleration greater than g'. We will find the optimal transition of the body from the state v_y^o, $v_{c\varphi}^o$, v_ϑ^o, r_H, ϑ_0 to the given circular trajectory of radius r_K. In the equations of motion (3.1) we may delete the projections of principal vector of carioliso forces. Since under slow conditions, the forces induced by nonuniform motion of particles are small, we may not confine to the condition m = const.

Equations (3.1) can be written as

$$\dot{v}_y = (v_\varphi^2 + v_\vartheta^2)(r_H + y)^{-1} - gr_H^2 (r_H + y)^{-2} - \dot{\nu} u_r \cos\psi,$$

$$\dot{v}_\varphi = v_\varphi (- v_y + v_\vartheta \operatorname{tg}\theta)(r_H + y)^{-1} - \dot{\nu} u_r \sin\psi \cos\gamma, \qquad (3.16)$$

$$\dot{v}_\vartheta = - (v_y v_\vartheta + v_\varphi^2 \operatorname{tg}\theta)(r_H + y)^{-1} - \dot{\nu} u_r \sin\psi \sin\gamma,$$

$$\dot{y} = v_y, \quad \dot{\vartheta} = v_\vartheta (r_H + y)^{-1}.$$

We will minimise the function in para 3 of section 2 in terms of maximum principle. Poutryagins function of our problem

$$H = \dot{\nu}(1 - \lambda_1 u_r \cos\psi - \lambda_2 u_r \sin\psi\cos\gamma - \lambda_3 u_r \sin\psi\sin\gamma) +$$

$$+ \lambda_1 \left[(v_\varphi^2 + v_\vartheta^2)(r_H + y)^{-1} - gr_H^2 (r_H + y)^{-2}\right] +$$

$$+ \lambda_2 v_\varphi (- v_y + v_\vartheta \operatorname{tg}\theta)(r_H + y)^{-1} - \lambda_3 (v_y v_\vartheta + v_\varphi^2 \operatorname{tg}\theta)(r_H + y)^{-1} +$$

$$+ \lambda_4 v_y + \lambda_5 v_\vartheta (r_H + y)^{-1} \qquad (3.17)$$

must be regarded as Hamiltonian function with impulses λ_i and coordinates v_y, v_φ, y, ϑ.

Thus we obtain the following Eulers equations.

$$\dot{\lambda}_1 - \lambda_2 v (r_H + y)^{-1} - \lambda_3 v_\vartheta (r_H + y)^{-1} + \lambda_4 = 0,$$

$$\dot{\lambda}_2 + 2\lambda_1 v_\varphi (r_H + y)^{-1} + \lambda_2 (- v_y + v_\vartheta \operatorname{tg}\theta)(r_H + y)^{-1} -$$

$$- 2\lambda_3 v_\varphi \operatorname{tg}\vartheta (r_H + y)^{-1} = 0,$$

$$\lambda_3 + 2\lambda_1 v_\theta (r_H + y)^{-1} + \lambda_2 v_\varphi \operatorname{tg}\theta (r_H + y)^{-1} - \lambda_3 v_y (r_H + y)^{-1} +$$

$$+ \lambda_5 (r_H + y)^{-1} = 0, \qquad (3.18)$$

$$\lambda_4 - \lambda_1 (v_\varphi^2 + v_\theta^2)(r_H + y)^{-2} + 2\lambda_1 g r_H^2 (r_H + y)^{-3} +$$

$$+\lambda_2 v_\varphi (v_y - v\theta \operatorname{tg}\theta)(r_H + y)^{-2} + \lambda_3 (v_y v_\theta + v_\varphi^2 \operatorname{tg}\theta)(r_H + y)^{-2}$$

$$-\lambda_5 v_\theta (r_H + y)^{-2} = 0,$$

$$\lambda_5 + \lambda_2 v_\varphi v_\theta (r_H + y)^{-1} \cos^{-2}\theta - \lambda_3 v_\varphi^2 (r_H + y)^{-1} \cos^{-2}\theta = 0.$$

Equations (3.16) and (3.18) must be solved together with steady boundary conditions: $t = 0$, $v_y = v_y^o$, $v_\varphi = v_\varphi^o$,

$v_\theta = v_\theta^o$, $y^o = 0$, $\theta = \theta^o$; $t = T$, $v_y^T = 0$, $v_\varphi^T = \sqrt{g r_H r_K}^{-\frac{1}{2}}$, $v_\theta^T = 0$,

$$y^T = r_K - r_H, \quad \theta^T = 0.$$

The function H in variables ν, ψ, γ attains a miaximum on optimal trajectory [80].

For internal extemum, provided such a point exists, we have

$$\frac{\partial H}{\partial \nu} = 1\, u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi \cos\gamma + \lambda_3 \sin\psi \sin\gamma) = 0,$$

$$\frac{\partial H}{\partial \psi} = \nu\, u_r (\lambda_1 \sin\psi - \lambda_2 \cos\psi \cos\gamma - \lambda_3 \cos\psi \sin\gamma) = 0,$$

$$\frac{\partial H}{\partial \gamma} = \nu\, u_r (\lambda_2 \sin\gamma - \lambda_3 \cos\gamma) \sin\psi = 0.$$

Compute the second derivatives of H

$$\frac{\partial^2 H}{\partial \dot{v}^2} = 0, \frac{\partial^2 H}{\partial \dot{v} \partial \psi} = u_r (\lambda_1 \sin\psi - \lambda_2 \cos\psi\cos\gamma - \lambda_3 \cos\psi \sin\gamma),$$

$$\frac{\partial^2 H}{\partial \dot{v} \partial \gamma} = u_r (\lambda_2 \sin\gamma - \lambda_3 \cos\gamma -) \qquad \sin\psi, \frac{\partial^2 H}{\partial \psi^2} =$$

$$= \dot{v}\, u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi \cos \; + \lambda_3 \sin\psi \sin\gamma),$$

$$\frac{\partial^2 H}{\partial \psi \partial \gamma} = \dot{v}\, u_r (\lambda_2 \sin\gamma - \lambda_3 \cos\gamma) \cos\psi, \frac{\partial^2 H}{\partial \gamma^2} =$$

$$= v\, u_r (\lambda_2 \cos\gamma + \lambda_3 \sin\gamma) \sin\psi.$$

The necessary and sufficient con itions for the maximality of H show that the extremal may consist of the parts of following trajectories:

Free motion if

$$\dot{v} = 0, \; 1 - u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi\cos\gamma + \lambda_3 \sin\psi \sin\gamma) \geqslant 0; \qquad (A)$$

Motion with variable reaction acceleration,

$$\lambda_1 = u_r^{-1} \cos\psi, \lambda_2 = u_r^{-1} \sin\psi\cos\gamma, \lambda_3 = u_r^{-1} \sin\psi \sin\gamma; \quad (B)$$

Motion with greatest reaction acceleration g',

$$\dot{v} = - u_r^{-1} g', \; 1 - u_r (\lambda_1 \cos\psi + \lambda_2 \sin\psi\cos\gamma + \lambda_3 \sin\psi\sin\gamma) < 0,$$

$$\lambda_1 \sin\psi - \lambda_2 \cos\psi\cos\gamma - \lambda_3 \cos\psi\sin\gamma = 0, \qquad (V)$$

$$(\lambda_2 \sin\gamma - \lambda_3 \cos\gamma) \sin\psi = 0.$$

In order that any part of trajectories (A), (B) or (V) may be included in optimal trajectory it is to be seen that equations (3.16) and (3.18) satisfy the trajectory referred to.

Integral H = h = const. must be satisfied along the optimal trajectory. If the maneuver time T is not known then h = 0.

On the trajectory (B), the above integral has the form

$$H = u_r^{-1}\left\{\left[(v_\varphi^2 - v_\theta^2)(r_H + y)^{-1} - gr_H^2(r_H + y)^{-2}\right]\cos\psi + \right.$$

$$+ (v_y v_\theta - v_\varphi^2 \operatorname{tg}\theta)(r_H + y)^{-1}\sin\psi\sin\gamma + v_y\dot{\psi}\sin\psi +$$

$$\left. + v_\theta\dot{\psi}\cos\psi\sin\gamma - v_\theta\dot{\gamma}\sin\psi\cos\gamma\right. = h. \qquad (3.19)$$

Formula (3.19) shows, that if the optimum trajectory ends at the planned sector (B), then as a result of boundary conditions, a change to a circular orbit h = 0 will also take place at a given time motion T.

We will use integral (3.19) for checking approximate optimal conditions.

In para 4, we will derive approximate initial conditions for the case (B) satisfying the equations (3.16) and (3.18) and the boundary conditions. These conditions may, in particular, include among them the sectors of free motion of the largest reaction acceleration and largest reaction force.

Therefore, these conditions completely solve the problem.

$$\xi = \sqrt{g r_H}\ (r_H + y)^{-\frac{1}{2}} - v_\varphi .$$

System (3.16) assumes the form

$$\dot{v}_y = -2\sqrt{g}\,\xi r_H\,(r_H + y)^{-\frac{3}{2}} + \xi^2\,(r_H + y)^{-1} + v_\theta^2\,(r_H + y)^{-1} - \nu u_r \cos\psi,$$

$$\dot{\xi} = \frac{1}{2}\sqrt{g} v_y r_H\,(r_H + y)^{-\frac{3}{2}} - \xi(v_y - v_\theta\,\mathrm{tg}\,\theta)\,(r_H + y)^{-1}$$

$$- \sqrt{g} v_\theta r_H\,\mathrm{tg}\,\theta\,(r_H + y)^{-\frac{3}{2}} + \nu u_r \sin\psi\cos\gamma,$$

$$\dot{v}_\theta = -v_y v_\theta\,(r_H + y)^{-1} + 2\sqrt{g}\,\xi r_H\,\mathrm{tg}\,\theta\,(r_H + y)^{-\frac{3}{2}} - \xi^2\,\mathrm{tg}\,\theta\,(r_H + y)^{-2} -$$

$$- g r_H^2\,\mathrm{tg}\,\theta\,(r_H + y)^{-2} - \nu u_r \sin\psi\sin\gamma, \qquad (3.20)$$

$$\dot{y} = v_y,\ \dot{\theta} = v_\theta\,(r_H + y)^{-1}.$$

Remove λ_4 and λ_5 from the system (3.18) and using the value of the equations (3.18) is replaced by the system

$$\dot{\lambda}_1 - 2\lambda_3 v\,(r_H + y)^{-1} + \lambda_1\left[g r_H^2\,(r_H + y)^{-1} - 6\sqrt{g}\,\xi r_H\,(r_H + y)^{-\frac{1}{2}} + \right.$$

$$\left. + \xi^2 - v_\theta^2\right](r_H + y)^{-2} + 2\lambda_3\,(v_y v_\theta - v_\varphi^2\,\mathrm{tg}\,\theta)\,(r_H + y)^{-2} +$$

$$+ \dot{v}u_r\,(\lambda_2 \cos\gamma + \lambda_3 \sin\gamma)\,(r_H + y)^{-1} \sin\psi = 0,$$

$$2 + 2\lambda_1 \left[\sqrt{g r_H}\,(r_H + y)^{-\frac{1}{2}} - \xi \right] (r_H + y)^{-1} +$$

$$+ \lambda_2 (-v_y + v_\theta \operatorname{tg}\theta)\,(r_H + y)^{-1} - 2\lambda_3 v_\varphi \operatorname{tg}\theta\,(r_H + y)^{-1} = 0, \quad (3.21)$$

$$\dot{\lambda}_3 = 2\lambda_1 v_\theta\,(r_H + y)^{-1} - 2\lambda_1 (v_y v_\theta + 2v_\varphi^2 \operatorname{tg}\theta)\,(r_H + y)^{-2} +$$

$$+ \lambda_3 \left[g r_H^2 (1 + 3\operatorname{tg}^2\theta)\,(r_H + y)^{-1} - 6\sqrt{g}\,\xi r_H \operatorname{tg}^2\theta\,(r_H + y)^{-\frac{1}{2}} + \right.$$

$$\left. + \xi^2 \operatorname{tg}^2\theta - v_\theta^2 \right] (r_H + y)^{-2} - \dot{v}u_r\,(2\lambda_1 \sin\psi \sin\gamma +$$

$$+ \lambda_2 \operatorname{tg}\theta \sin\psi \cos\gamma - \lambda_3 \cos\psi)\,(r_H + y)^{-1} = 0.$$

We note that the quantities g, $\dot{v}u_r$, v_y, ξ, $v\dot{\theta}$, $r_K - r_H$, $r_H\theta$ are of zero order. Hence ϑ has order r_H^{-1}.

4. Approximate Optimal Controls.

Suppose the transition of the body on the circular trajectory is quite rapid so that the terms of order $r_H^{-1/2}$ do not attain relatively high values during the motion time T.

The system (3.20) precisely upto terms of order $r_H^{-1/2}$ is written in the form

$$\dot{v}_y = -\dot{v}u_r \cos\psi, \quad \dot{\xi} = \dot{v}u_r \sin\psi\cos\gamma,$$

$$\dot{v}_\theta = -\dot{v}u_r \sin\psi \sin\gamma, \quad \dot{y} = v_y, \quad r_H\dot{\vartheta} = v_\vartheta\ . \quad (3.22)$$

Equations (3.21) precisely upto terms of order $r_H^{-1/2}$ are written as

$$\lambda_1 = \lambda_1^0 + a_1 t, \quad \lambda_2 = \lambda_2^0, \quad \lambda_3 = \lambda_3^0 + a_3 t,$$

where a_1 and a_3 are constents. For programed conditions (B) we have

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = u_r^{-2} \tag{3.23}$$

This gives $a_1 = 0$, $a_3 = 0$, $\psi = \psi_0 = \text{const}$, $\gamma = \gamma_0 = \text{const}$.

Steadiness of the angles of inclination of reaction force and the formulea (3.22) show that precisely upto terms of the order $r_H^{-1/2}$ we may use formulae for sharp optimal manoeuvre (3.14) and (3.15).

Satisfying the boundary conditions in y and , we get the conditions

$$\begin{aligned} v_y^0 T - u_r \cos\psi_0 \int_0^T (\nu - \nu_0)\, dt &= r_K - r_H + 0\,(r_H^{-\frac{1}{2}}), \\ v_\theta^0 T - u_r \sin\psi_0 \sin\gamma_0 \int_0^T (\nu - \nu_0)\, dt &= - r_H \theta_0 + 0\,(r_H^{-\frac{1}{2}}), \end{aligned} \tag{3.24}$$

which together with (3.14) must be satisfied by the law of change of mass $\nu(t)$ and the time of motion T.

Time of motion is given by

$$T = \frac{(r_K - r_H)\sin\psi_0 \sin\gamma_0 + r\vartheta_0 \cos\psi_0}{v_y^0 \sin\psi_0 \sin\gamma_0 - v_\vartheta^0 \cos\psi_0},$$

which substitutes one of the conditions (3.24). Angles of inclination of reaction force are determined by (3.15).

Thus obtained approximate optimal conditions satisfy relation (3.19) when h = 0. Actually the function H as used in (3.19) turns out to be equal to $H = 0 + O(r_H^{-\frac{1}{2}} u_r^{-1})$.

Example 1. Let the law of change of mass be taken as

$$m = m_0 e^{-(\alpha t + \beta t^2)}.$$

Finding the constants α and β such that they satisfy the boundary conditions, we have

$$\alpha T + \beta T^2 = v_0 - v_T = \frac{V}{u_r},$$

$$\frac{\alpha T^2}{2} + \frac{\beta T^3}{3} = \frac{r_K - r_H - v_y^0 T}{u_r \cos\psi_0}.$$

The constants are thus equal to

$$\alpha = 6u_r^{-1} T^{-2} \left[(r_K - r_H - v_y^0 T)\cos^{-1}\psi_0 - \frac{1}{3} VT\right],$$

$$\beta = 6u_r^{-1} T^{-3} \left[-(r_K - r_H - v_y^0 T)\cos^{-1}\psi_0 + \frac{1}{2} VT\right].$$

Reaction acceleration can not be greater than g'. Therefore, the specified law of combustion can be taken in the case when

$$g' \geq u_r \, (\alpha + 2\beta T).$$

Example 2. Consider the conditions including the trajectory of largest reaction acceleration g' and free motion. In the interval $[0, \tau_1)$ We have $\nu = \nu_0 - g' u_r^{-1} t$, in the interval $[\tau_1, \tau_2)$ let there be free motion and $\nu = \nu_0 - g' u_r^{-1} \tau_1$ and in the interval $[\tau_2, T]$ $\nu = \nu_0 - g' u_r^{-1} \tau_1 - g' u_r^{-1} (t - \tau_2)$. τ_1 and τ_2 are determined in such a way that the boundary conditions are satisfied.

We have the conditions

$$T + (T - \tau_2) - (T - \tau_1) = V (g')^{-1},$$

$$T^2 + (T - \tau_2)^2 - (T - \tau_1)^2 = 2 (r_K - r_H - v_y^o T) (g' \cos \psi_o)^{-1}.$$

Thus we get

$$\tau_1 = \frac{1}{2} T - \left[2 (r_K - r_H - v_y^o T) - g' T^2 \cos \psi_o\right] \times$$
$$(V - g'T)^{-1} \cos^{-1} \psi_o + V (g')^{-1} \Big],$$

$$\tau_2 = \frac{1}{2} \Big\{ 3T - \left[2 (r_K - r_H - v_y^o T) - g' T^2 \cos \psi_o\right] \times$$
$$\times (V - g'T)^{-1} \cos^{-1} \psi_o - V (g')^{-1} \Big\}.$$

If the quantities v_y^o, ξ_o, v_θ^o, $r_K - r_H$, $r_K \theta_o$, are relatively large then we may get approximate optimal manoeuvre with increased accuracy.

Equations (3.20) precisely upto terms of order r_H^{-1} are written as

$$\dot{v}_y = -2\Omega\xi - \nu u_r \cos\psi,$$

$$\dot{\xi} = \frac{1}{2}\Omega v_y + \nu u_r \sin\psi\cos\gamma, \qquad (3.25)$$

$$\dot{v}_\theta = -\nu u_r \sin\psi\sin\gamma,$$

$$\dot{y} = v_y, \; r_H\theta = v_\theta .$$

Here $\Omega = \sqrt{g r_H r_K^{-\frac{3}{2}}}$ is the angular velocity of the body on circular trajectory. This is of the order of $r_H^{-1/2}$.

Omitting the terms of the order of r_H^{-1}, in accordance with the system (3.21) we get

$$\ddot{\lambda}_1 = 0, \; \ddot{\lambda}_2 + 2\Omega\dot{\lambda}_1 = 0, \; \ddot{\lambda}_3 = 0.$$

This gives

$$\lambda_1 = \lambda_1^0 + a_1 t, \; \lambda_2 = \lambda_2^0 - 2\Omega\lambda_1^0 t - 2\Omega\lambda_1^0 t - \Omega a_1 t^2, \; \lambda_3 = \lambda_3^0 + a_3 t$$

In accordance with (3.23) we find

$$a_1 = 2\lambda_2^0\Omega, \; a_3 = 0.$$

Angles of inclination of the pull in view of formula (B) are determined in the form

$$\psi = \psi_0 - 2\Omega t\cos\gamma_0 + 0(r_H^{-1}),$$

$$\gamma = \gamma_0 + 2\Omega t \operatorname{ctg}\psi_0 \sin\gamma_0 + 0(r_H^{-1}). \qquad (3.26)$$

For the sake of simplicity, let v° , $r_H\,\theta_o$, γ be of order $r_H^{-1/2}$.

We obtain a solution of equations (3.25) when the angles of inclination ψ and γ are given by (3.26) with an accuracy upto $O(r_H^{-1})$.

$$v_y = v_y^o - u_r\,(\nu - \nu_o)\cos\psi_o - 2\Omega\left[\xi_o + u_r\,(\nu - \nu_o)\sin\psi_o\cos\gamma_o\right]t,$$

$$\xi = \xi_o + u_r\,(\nu - \nu_o)\sin\psi_o\cos\gamma_o + \frac{1}{2}\Omega\,v_y^o t +$$

$$+ \frac{3}{2}\Omega\,u_r\cos\psi_o\int_o^t (\nu - \nu_o)\,dt - 2\Omega u_r\,(\nu - \nu_o)\,t\cos\psi_o,$$

$$v = v_\theta^o - u_r\,(\nu - \nu_o)\sin\psi_o\sin\gamma_o, \qquad (3.27)$$

$$y = v_y^o t - \Omega\,\xi_o t^2 - u_r\cos\psi_o\int_o^t (\nu - \nu_o)\,dt -$$

$$- 2\Omega u_r\sin\psi_o\cos\gamma_o\int_o^t (\nu - \nu_o)\,dt,$$

$$r_H\theta = r_H\,\theta_o - v_\theta^o\,t - u_r\sin\psi_o\sin\gamma_o\int_o^t (\nu - \nu_o)\,dt.$$

The boundary conditions are $v_y^T = 0$, $\xi_T = 0$, $v_\theta^T = 0$, $y_T = r_K - r_H$, $r_H\,\theta_T = 0$.

$$\int_o^T (\nu - \nu_o)\,dt = -\,(r_K - r_H - v_y^o T)\,(u_r\cos\psi_o)^{-1} + O\left(r_H^{-\frac{1}{2}}\right).$$

Therefore, in accordance with first three expressions in the system (3.27) we find

$$v_y^o - 2\Omega\,\xi_o T = u_r\,(\nu_T - \nu_o)\,(\cos\psi_o + 2\Omega T\sin\psi_o\cos\gamma_o),$$

$$\xi_0 + 2\Omega v_y^0 T - \frac{3}{2}\Omega (r_K - r_H) =$$

$$= u_r (\nu_T - \nu_0)(-\sin\psi_0 \cos\gamma_0 + 2\Omega T \cos\psi_0),$$

$$v_\theta^0 = u_r (\nu_T - \nu_0) \sin\psi_0 \sin\gamma_0. \qquad (3.28)$$

Squaring the relations (3.28) and adding, we get the expression for mass at the time T

$$m_T = m_0 \exp\left\{-V u_r^{-1}\left[1 - \frac{3}{2}\Omega\xi_0 V^{-2}(r_K - r_H)\right]\right\}. \qquad (3.29)$$

Put in (3.28) $\psi_0 = \psi'_0 + \psi''_0$, $\gamma_0 = \gamma'_0 + \gamma''_0$, where ψ'_0 and γ'_0 are found from (3.15). We will then have the corrections for initial angles of inclination of reaction force

$$\psi'' = \frac{3\Omega v_y^0}{2V^2}(r_K - r_H), \quad \gamma'' = 0\,(r_H^{-1}).$$

The last two equations of the system (3.27) precisely upto terms of the order of r_H^{-1} give the conditions

$$u_r \cos\psi_0 \int_0^T (\nu - \nu_0)\,dt - 2\Omega u_r \sin\psi_0 \cos\gamma_0 \int_0^T \int_0^t (\nu - \nu_0)dt -$$

$$- v_y^0 T (1 - 2\Omega T \operatorname{tg}\psi_0 \cos\gamma_0) + \Omega\xi_0 T^2 +$$

$$+ (r_K - r_H)(1 - 2\Omega T \operatorname{tg}\psi_0 \cos\gamma_0) = 0, \qquad (3.30)$$

$$u_r \sin\psi_0 \sin\gamma_0 \int_0^T (\nu - \nu_0)\,dt + v_\theta^0 T - r\theta_0 = 0.$$

The conditions (3.29) and (3.30) will be satisfied by T and the law of change of mass in the same way as the conditions (3.14) and (3.24) did. In particular, we may take the laws of change of mass as considered in the above examples 1 and 2. The formula (3.29) shows that the consumption of mass does not depend on the law of change of mass.

The function H and employed in (3.19) in accordance with formulae (3.26) and (3.27) is equal to

$$H = - 2\Omega u_r^{-1} (\xi_o \cos \psi'_o + v_y^o \sin \psi'_o \cos \gamma'_o) + O (r_H^{-1} u_r^{-1}).$$

Trigonometric functions of the angles ψ'_o and γ'_o are determined by the formulae (3.15) and hence $H = O + O (r_H^{-1} u_r^{-1})$.

CHAPTER VIII

Presently there is wide application of the motion of bodies in air and vaccum which is produced by the discharge of mass. It is anticipated that in future this principle will be widely used for producing motions on earth and other planets. It may be stressed that experiments will have to be carried out using reaction motors for the development of automobiles and aero-floats.

It will be of interest to study the possibility of developing mono-wheel transport using sledges and automobiles with variable masses.

1. Investigations about the Stability of Mano-wheel Carriage with variable mass.

1. Setting out the problem. The main problem to be considered in this connection is the tilting of mano-wheel carriage with variable mass on the horizontal plane [69] taking into account the internal motion of particles and different kinds of resistances. Stability against tilting will be ensured by moving forward the supporting wheel 6 (Figure 14). The uniform straightline motion of the carriage will be asymptotically stable if we use ideal autopilot which will deflect the air or gas rudders proportional to the angles of inclination and turning of the plane of working wheels.

The mono-wheel carriage with variable mass (figure 14) is made of working wheel 1 with constant mass and a double passenger cradle 2, freely rotating on axes and carrying on its reaction motors 3, gas rudders 4, air rudders 5 and the fuel tank. System (ξ, η, ζ) is fixed (O, ξ', η', ζ') is parallel to the system (ξ, η, ζ), the system (O, x, y, z) brings the wheel 1 into motion but is not responsible for its own gyration. The axes ζ and $O\zeta'$ are in vertical direction, the axes Ox and Oy lie in the plane of the working wheel, axis Ox is directed towards the point of contact D, axis Oy is horizontal and parallel to the line KL which is the line of intersection of the plane of the wheel and the horizontal plane.

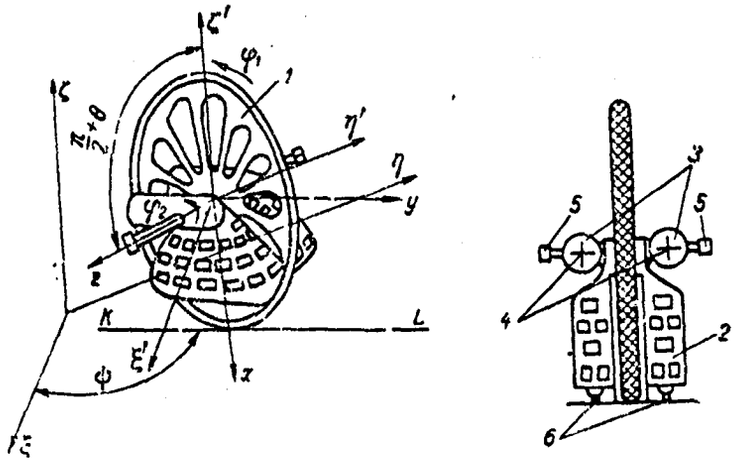

Рис. 14

Figure - 14.

2. Equations of motion. The motion of the carriage is determined by six coordinates: two coordinates ξ, η of the point of contact, three Eulerian angles of working wheel φ_1, ψ, θ and the tilting angle φ_2 of the cradle. These coordinates are related by

two non-holonomic relations.

$$\dot{\xi} = -a\dot{\varphi}_1 \cos\psi, \quad \eta = -a\dot{\varphi}_1 \sin\psi.$$

Denote the angular velocity of rotation of the system (O, x, y, z) by $\bar{\omega}$. Then the angular velocities of working wheel and the cradle are respectively equal to $\bar{\omega}_1 = \bar{\omega} + \dot{\varphi}_1 \bar{k}$, $\bar{\omega}_2 = \bar{\omega} + \dot{\varphi}_2 \bar{k}$ where $\bar{k}$ is orthognal to Oz. Projecting on x, y, z axes, we have

$$\omega_{1x} = \omega_{2x} = \omega_x, \quad \omega_x = -\dot{\psi}\cos\theta,$$

$$\omega_{1y} = \omega_{2y} = \omega_y, \quad \omega_y = \theta,$$

$$\omega_{1z} = \omega_z + \dot{\varphi}_1, \quad \omega_{2z} = \omega_z + \dot{\varphi}_2, \quad \omega_z = -\dot{\psi}\sin\theta. \tag{1.1}$$

Vectorial equations of motion of working wheel are written as

$$I_1 \dot{\bar{\omega}}_1 + \bar{\omega} I_1 \bar{\omega}_1 = \bar{r}_D \times \bar{S} + \bar{L}_S + \bar{L}_T + \bar{L}_{W1}, \tag{1.2}$$

$$m_1 (\dot{\bar{v}}_o + \bar{\omega} \times v_o) = m_1 \bar{g} + \bar{S} + \bar{T} + \bar{W}_1, \tag{1.3}$$

Where m_1 is the mass of wheel.

$$I_1 = \begin{Vmatrix} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & C_1 \end{Vmatrix}$$

is inertial tensor of the wheel in the system (O, x, y, z); $\bar{v}$ - velocity of the center O; $\bar{r}_D$ - radius vector of the point D with respect to O; $\bar{S}$ - reaction of the plane at point D; $\bar{T}$ - principal vector of the reaction one the axis Oz; W_1 - resistance of air when the wheel moves,

L_S, L_T and L_{W_1} - they are respectively moments of rolling friction, reaction forces on Oz and air resistance, dash[1] on the head denotes derivative in the system (O, x, y, z).

Vectorial equations of the cradle which is regarded as a body of variable mass (see 6, Chapter II) are written as

$$(I_2\bar{\omega}_2)'^* + \bar{\omega} \times I_2\bar{\omega}_2 + m_2\bar{r}_C \times (\bar{v}_O + \bar{\omega} \times \bar{v}_O) =$$

$$= m_2\bar{r}_C \times \bar{g} - \bar{L}_T + \bar{L}_{W2} + \bar{M}_O, \tag{1.4}$$

$$m_2\left[v_O + \bar{\omega} \times \bar{v}_O + \bar{\omega}_2' \times \bar{r}_C + \bar{\omega}_2 \times \bar{r}_C'^* + \bar{\omega} \times (\bar{\omega}_2 \times r_C)\right] =$$

$$= m_2\bar{g} - \bar{T} + \bar{W}_2 + \bar{R}, \tag{1.5}$$

where m_2 is the mass of cradle;

$$I_2 = \begin{Vmatrix} A_2\cos^2\varphi_2 + B_2\sin^2\varphi_2 & (A_2 - B_2)\sin\varphi_2\cos\varphi_2 & 0 \\ (A_2 - B_2)\sin\varphi_2\cos\varphi_2 & A_2\sin^2\varphi_2 + B_2\cos^2\varphi_2 & 0 \\ 0 & 0 & C_2 \end{Vmatrix}$$

is inertial tensor of cradle in system (O, x, y, z); A_2, B_2, C_2 - principal moments of inertia; $\bar{r}_C$ - radius vector of center of gravity of cradle. As a result of change of mas, the quantities m_2, A_2, B_2, C_2 and $\bar{r}_C$ are regarded as functions of time. However, it may be noted that while computing the derivative with star, masses remain fixed.

In equations (1.4) and (1.5) we have $\bar{W}_2$ - air resistance when the cradle is in motion, $\bar{R}$ - reaction force; $\bar{L}_{W_2}$ and $\bar{M}$ - they are respectively moments of air resistance and reaction force. All moments are calculated about the point O.

Add equations (1.2) and (1.4); also add equations (1.3) and (1.5). Removing from these sums the reaction $\bar{R}$, we have

$$\left[(I_1 + I_2)\bar{\omega}\right]' + (C_1\dot{\varphi}_1 + C_2\dot{\varphi}_2)\,k + \bar{\omega} \times \left[(I_1 + I_2)\bar{\omega} +$$

$$+ (C_1\varphi_1 + C_2\varphi_2)\,\bar{k}\right] + m_2\left[\bar{r}_C \times (\bar{v}_o + \bar{\omega} \times \bar{v}_o)\right] - (m_1 + m_2)\,\bar{r}_D \times$$

$$\times (\bar{v}_0 + \bar{\omega} \times \bar{v}_o) - m_2\bar{r}_D \times \left[\bar{\omega}_2 \times \bar{r}_C + \bar{\omega}_2 \times \bar{r}_C' + \bar{\omega} \times (\bar{\omega}_2 \times \bar{r}_C)\right] =$$

$$= m_2\bar{r}_C \times \bar{\varepsilon} - (m_1 + m_2)\,\bar{r}_D \times \bar{g} + \bar{L}_S + \bar{L}_{W_1 D} + \bar{L}_{W_2 D} + \bar{M}_D. \qquad (1.6)$$

Index D means that the moments have been found about the point D.

Suppose that the mass changes steadily and the full supply is symmetrical about the two axes of symmetry of cradle. Then the reaction moment according to formulas 2, Chapter V is given by

$$\bar{M}_D = (-\nu_1^2\omega_x + r_D Q_y^r\omega_y, -\nu_2^2\omega_y - r_D Q_y^r\omega_x, l_2^r - l_1^r - \nu_3^2\omega_{2z}), \qquad (1.7)$$

Where ν_1^2, ν_2^2, ν_3^2 are vector function of time arising from carioliso moment of particles, l_1^r and l_2^r are respectively the per second input and output of kinetic moments of particles in reaction motors. These moments are about the point D. Q_y^r is projection of relative momentum of particles on axis Oy. We will have have $Q_y^r = Q_1 \cos\varphi_2 + Q_2 \sin\varphi_2$, where Q_1 = const and Q_2 = const.

3. Construction of Lyapuov functions. We will suppose that l_1^r and l_2^r are automatically regulated in such a way that there is a uniform tilting of working wheel i.e. $\dot{\varphi}_1 = \text{const}$.

Projecting the equation (1.4) on Oz axis and restricting to terms of first order infinitesimal we get

$$C_2 \dot{\varphi}_2 + (k^2 + \nu^2)\varphi_2 + m_2 g r_C \varphi_2 = 0. \tag{1.8}$$

Here k^2 - is the constant of air resistance; $\nu^2 \varphi_2$ is projection of carioliso moment of particles about the point O on Oz axis; C_2 and $m_2 r_C$ are positive and decreasing functions of time.

Set up a positive definite function

$$V_1 = \frac{1}{2}(C_2 \varphi_2^2 + m_2 g r_C \varphi_2^2). \tag{1.9}$$

In accordance with (1.8), derivative of V_1 will be negative and can be put in the following form

$$\frac{dV_1}{dt} = -(k^2 + \nu^2)\varphi_2^2 + \frac{1}{2}\left[C_2 \dot{\varphi}_2^2 + (m_2 r_C)\cdot g \varphi_2^2\right].$$

It is, therefore, easy to see that as mass changes φ_2 and $\dot{\varphi}_2$ decrease exponentially.

Projecting equation (1.6) on the axes Ox and Oy and restricting to terms of first order infinitesimal in respect ov variables φ_2, , and we have

$$(A_1 + A_2)\dot{\psi} + (k_1^2 + \nu_1^2)\psi - C_1 \dot{\varphi}_1 \theta + b_1^2 \psi = 0, \tag{1.10}$$

$$(\tilde{A}_1 + \tilde{B}_2)\dot{\theta} + \left[\tilde{C}_1 + m_2 r_D (R_D - r_C)\right] \dot{\varphi}_1 \dot{\psi} + (k_2^2 + \nu_2^2)\theta +$$

$$+ \left[b_2^2 - m_1 r_D - m_2 (r_D - R_C)\right] \theta = 0, \qquad (1.11)$$

where

$\widetilde{A}_1 = A_1 + m_1 r_D^2$, $\widetilde{B}_2 + m_2 r_D^2 - 2m_2 r_C r_D$, $\widetilde{C}_1 = C_1 + m_1 r_D^2$

are corresponding inertial moments about the axes passing through the point D as origin, D; b_1^2 and b_2^2 are constants of autopilot.

Set up a positive definite function of Lyapumov

$$V_2 = \frac{1}{2}\left[\widetilde{C}_1 + m_1 r_D (r_D - r_C)\right]\left[(A_1 + A_2)\psi^2 + b_1^2 \psi^2\right] +$$

$$+ \frac{1}{2} C_1 (\widetilde{A}_1 + \widetilde{B}_2)\theta^2 + \frac{1}{2} C_1 \left[b_2^2 - m_1 r_D - m_2 (r_D - r_C)\right] \theta^2. \qquad (1.12)$$

Suppose $b_2^2 > m_1 r_D + m_2 (r_D - r_C)$. If the functions of time ψ, $\dot{\psi}$ and θ, $\dot{\theta}$ are non-increasing then in consonance with the equations (1.10) and (1.11) dV_2/dt will be negative and definite.

From the functions (1.9) and (1.12) it follows that Lyapvov characteristic numbers are positive for the equations (1.8), (1.10) and (1.11). If this system of equations is consistent [48] then the non-linear system of equations dealing with the tilt of reactive carriage on the horizontal plane will be asymptotically stable according to Lyapuov.

If the reactive carriage moves on the surface of a planet then $l_1^r = 0$, $k_1 = k_2 = k = 0$. Thus, as before uniform reactilinear motion will gain asympototic stability.

4. Stability of mono-wheel carriage with non converted autopilot. Suppose the autopilot constants b_1 and b_2 are equal to zero. Let us neglect the effect of resistance and the internal motion of particles. Equations (1.10) and (1.11) are written in the form

$$\dot{\psi} - \beta\dot{\theta} = 0,$$
$$\theta + \tilde{\beta}\psi = \alpha\theta. \qquad (1.13)$$

In these

$$\beta = \frac{C_1\varphi_1}{A_1 + A_2}, \quad \tilde{\beta} = \frac{\tilde{C}_1 + m_2 r_D (r_D - r_C)}{\tilde{A}_1 + \tilde{B}_2} \qquad (1.14)$$

$$\alpha = \frac{m_1 r_D + m_2 (r_D - r_C)}{\tilde{A}_1 + \tilde{B}_2} \qquad (1.15)$$

α, β and $\tilde{\beta}$ are positive functions of time.

We will examine the system (1.13) for stability of zero solution.

Set up a function

$$V = p_1 (\theta - \lambda\psi)^2 + p_2\dot{\theta}^2 + p_3\omega^2,$$

where p_1, p_2, p_3 are certain function of time and

$$p_i(t) \geq d_i^2 = \text{const} \qquad (i = 1, 2, 3),$$

λ is some unknown constant.

If, according to equations (1.13), $\frac{dV}{dt} \leq 0$, then θ, $\dot{\theta}$ and ψ, stabilise because in that case

$$\leq \frac{1}{d_i} \sqrt{\frac{\{|\theta - \lambda\psi|, |\theta|, |\dot{\psi}|\} \leq}{p_1(0)(\theta_o - \lambda\psi_o)^2 + p_2(0)\theta_o^2 + p_3(0)\psi_o^2}}.$$

Further let

$$d_1 > 0,\ d_2 > 0,\ d_3 > 0.$$

Then

$$\frac{dV}{dt} = \dot{p}_1\theta^2 + \dot{p}_2\theta^2 + (\dot{p}_3 + \lambda^2\dot{p}_1)\dot{\psi}^2 +$$

$$+ 2\left[p_1(1 - \lambda\beta) + p_2\alpha\right]\theta\dot{\theta} - 2\dot{p}_1\lambda\theta\psi +$$

$$+ 2\left[-p_1\lambda(1 - \lambda\beta) - p_2\tilde{\beta} + p_3\beta\right]\theta\dot{\psi}.$$

In order to ensure that $\frac{dV}{dt} \leq 0$, it is sufficient that the following conditions are satisfied

$$\dot{p}_1 \leq 0,\ \dot{p}_2 \leq 0,\ \dot{p}_3 + \lambda^2\dot{p}_1 \leq 0, \tag{1.16}$$

$$\sqrt{-\dot{p}_1}\sqrt{-\dot{p}_2} \geq \left|p_1(1 - \lambda\beta) + p_2\alpha\right|, \tag{1.17}$$

$$\sqrt{-\dot{p}_1}\sqrt{-\dot{p}_3 - \lambda^2\dot{p}_1} \geq |\lambda|\dot{p}_1|, \tag{1.18}$$

$$\sqrt{-\dot{p}_2}\sqrt{-\dot{p}_3 - \lambda^2\dot{p}_1} \geq \left|-p_1\lambda(1 - \lambda\beta) - p_2\tilde{\beta} + p_3\beta\right|, \tag{1.19}$$

$$-\dot{p}_1 \qquad -\left[p_1(1 - \lambda\beta) + p_2\alpha\right]$$

$$-\left[p_1(1 - \lambda\beta) + p_2\alpha\right] \qquad -\dot{p}_2$$

$$\lambda\dot{p}_1 \qquad -\left[-p_1\lambda(1 - \lambda\beta) - p_2\tilde{\beta} + p_3\beta\right]$$

$$\dot{p}_1\lambda \ .$$

$$- \left[- p_1 \lambda (1 - \lambda \beta) - p_2 \tilde{\beta} + p_3 \beta \right] \geqslant 0. \quad (1.20)$$

$$- (\dot{p}_3 + \lambda^2 \dot{p}_1)$$

Let us put

$$p_2 = \gamma p_1, \quad p_3 = \lambda (\sigma - \lambda) p_1,$$

where γ - is some positive constant; σ is constant; while $\lambda (\sigma - \lambda) > 0$. Hence $\lambda_\sigma > 0$.

Condition (1.18) is satisfied while the remaining conditions assume the form

$$\dot{p}_1 \leqslant 0, \quad (1.21)$$

$$- \frac{\dot{p}_1}{p_1} \geqslant \frac{1}{\sqrt{\gamma}} \left| 1 - \lambda \beta + \mu \alpha \right|, \quad (1.22)$$

$$- \frac{\dot{p}_1}{p_1} \geqslant \frac{1}{\sqrt{\lambda \gamma \sigma}} \left[\lambda (\sigma \beta - 1) - \gamma \tilde{\beta} \right]. \quad (1.23)$$

Condition (1.20) remains unchanged.

Let the process of the change of mass of the carriage takes place for a finite time T, then when $t \geqslant T$, $\alpha, \beta, \tilde{\beta}$ are constant and equal to α_T, β_T, $\tilde{\beta}_T$.

and are chosen in such a way that the equations $1 - \lambda \beta_T + \gamma \alpha_T = 0$ and $\lambda (\sigma \beta_T - 1) - \gamma \tilde{\beta}_T = 0$ are satisfied

These are solved to find and

$$\lambda = \frac{\beta_T}{\beta_T \tilde{\beta}_T - \tilde{\sigma}\,\alpha_T}, \quad \gamma = \frac{\tilde{\sigma}}{\beta_T \tilde{\beta}_T - \tilde{\sigma}\,\alpha_T}, \qquad (1.24)$$

where

$$\tilde{\sigma} = \sigma \beta_T - 1.$$

The following condition

$$\beta_T \tilde{\beta}_T - \tilde{\sigma}\,\alpha_T \neq 0.$$

is necessary.

$\tilde{\sigma}$ must be greater than zero because in the opposite event $\gamma \leq 0$. Then the condition

$$\beta_T \tilde{\beta}_T - \tilde{\sigma}\,\alpha_T > 0. \qquad (1.25)$$

is satisfied.

$\lambda > 0$. Taking into account the condition

$$\lambda\,(\sigma - \lambda) > 0,$$

We conclude that $\sigma > \lambda > 0$. Therefore, according to (1.24) we have

$$\sigma > \frac{\tilde{\beta}_T}{\beta_T \tilde{\beta}_T - \tilde{\sigma}\,\alpha_T}$$

Thus there exists a small and positive number (as small as we please) such that the relation

$$\beta_T \tilde{\beta}_T - \alpha_T (1 + \tilde{\sigma}) > 0. \qquad (1.26)$$

is satisfied.

Note that condition (1.26) implies (1.25).

Condition (1.20) may be written as

$$-\frac{\dot{p}_1}{p_1} \geq \frac{}{\sqrt{\lambda\gamma(\sigma-\lambda)}}\sqrt{\lambda(\sigma-\lambda)(1-\lambda\beta+\gamma\alpha)^2+}$$

$$+\left[\lambda(1-\lambda\beta+\gamma\alpha)+\lambda(\sigma\beta-1)-\gamma\tilde{\beta}\right]^2.$$

Choose p_1, such that

$$-\frac{\dot{p}_1}{p_1} = \chi(t),$$

Where

$$\chi(t) = \frac{}{\sqrt{\lambda\gamma(\sigma-\lambda)}}\sqrt{\lambda(\sigma-\lambda)(1-\lambda\beta+\gamma\alpha)^2+}$$

$$+\left[\lambda\left[1-\lambda\beta+\gamma\alpha\right]+\left[\lambda(\sigma\beta-1)-\gamma\tilde{\beta}\right]\right]^2.$$

Then the conditions will be especially fulfilled (1.21),(1.23) & (1.26)

Suppose $p_1 = e^{-\int_0^t x(t)\,dt}$

Then we have

$$p_1 \geq e^{-\int_0^T x(t)\,dt} = d_1^2. \qquad (1.27)$$

The above discussion is summarized as follows. If mass changes only for a finite time then there exists a T such that whenever $t \geq T$,

$$\alpha = \alpha_T, \quad \beta = \beta_T, \quad \tilde{\beta} = \tilde{\beta}_T,$$

where α_T, β_T, $\tilde{\beta}_T$ are positive constants. Further, if there exists $\tilde{\sigma} > 0$ such that (1.26) is satisfied then the motion of mono-wheel carriage is stable and

Actually, suppose the following conditions are satisfied

$$\{|\theta_0|, \quad |\dot{\theta}_0|, \quad |\psi_0|\} < \delta,$$

then we may write

$$\{|\theta|, |\psi|\} < P d_i^{-1} \delta \quad (i = 2, 3), \qquad (1.28)$$

$$|\theta| \; P \; (d_1^{-1} + \; d_3^{-1}) \; , \qquad (1.29)$$

Where λ is determined by (1.24) and

$$P = \sqrt{\; p_1(0) \; (1 + \lambda)^2 + p_2 \; (0 + p_3(0).} \qquad (1.30)$$

In formulae (1.28) - (1.30) let us put

$$p_1(0) = 1, \quad p_2(0) = \frac{\tilde{\sigma}}{\beta_T \; \tilde{\beta}_T - \tilde{\sigma} \alpha_T}$$

$$p_3(0) = \lambda \left(\frac{1 + \tilde{\sigma}}{\tilde{\beta}_T} - \lambda\right), \; d_1^2 = e^{-\int_0^T x(t)\, dt},$$

$$d_2^2 = \frac{\tilde{\sigma} d_1^2}{\beta_T \; \tilde{\beta}_T - \tilde{\sigma} \alpha_T}, \; d_3^2 = \lambda \left(\frac{1 + \tilde{\sigma}}{\tilde{\beta}_T} - \lambda\right) d_1^2.$$

If the mass changes for infinite time, then we can state tne following about the stability. If there exist constants λ, γ and σ such that

$$\lambda(\sigma - \lambda) > 0, \; \gamma > 0$$

and the integral

$$\int_0^\infty \frac{\lambda(\sigma - \lambda)(1 - \lambda\beta + \gamma\alpha)^2 + [|\lambda(1 - \lambda\beta + \gamma\alpha) + |\lambda(\sigma\beta - 1) - \gamma]^2}{\sqrt{\lambda\gamma(\sigma - \lambda)}} dt$$

Converges then the motion would be stable.

5. Special Cases of Stability of Mono-wheel Carriage of Variable Mass.

Let us assume that a disc is used for the transportation of passangers. If, during the acceleration period the moment of reactive forces is great, then with stabilized motion, i.e. when the motion is sufficiently fast, we shall consider that the magnitude of the above moment is fairly small. Stability during acceleration, as we have already pointed out, can be secured by moving forward the corresponding supports 6 (see figure 14).

Consequently, we will think that φ, A and C are considerably small. In this case $\theta_0 = 0$ and $\psi_0 = 0$ and we may examine instead of the system (1.13) the stability of zero solution of the equation

$$\ddot{\vartheta} + s(t)\,\vartheta = 0, \qquad (1.31)$$

where

$$s(t) = \beta\beta - \alpha \qquad (1.32)$$

Physically stability means the stability against lateral jolts.

Consider the Lyapunov function

$$V = p(t)\,\vartheta^2 + q(t)\,\dot{\vartheta}^2.$$

Suppose the following conditions are satisfied

$$p(t) \geqslant d_1^2,\ q(t) \geqslant d_2^2,$$

$$\dot{p} \leqslant 0,\ \dot{q} \leqslant 0,$$

$$\sqrt{-\dot{p}}\,\sqrt{-\dot{q}} \geqslant \left| p - qs(t) \right|, \qquad (1.33)$$

Then we have

$$\frac{dV}{dt} \leqslant 0.$$

When $|\dot{\theta}_0| < \delta$ get

$$|\theta| \; \delta d_1^{-1} \sqrt{\gamma q(0)}, \qquad |\dot{\theta}| < \delta d_2^{-1} \sqrt{q(0)}. \tag{1.34}$$

Put $p(t) = 1$, the $q(t) = s^{-1}(t)$. Suppose $s(t) > 0$. Since $\dot{q} = -\dot{s}s^{-2}$, therefore it is necessary that we must have

$$\dot{s}(t) \geqslant 0.$$

Thus the zero solution of equation (1.31) will be stable if the function s(t) is finite positive and increasing.

Let $q(t) = 1$, then function p (t) must be equal to s(t). Suppose the conditions $\dot{s}(t) \leqslant 0$, $s(t) \geqslant d_1^2$ are satisfied. Consequently, the zero solution of (1.31) will be stable if the function s(t) is positive and decreasing but it does not become zero.

Note that the functions are not strictly increasing and decreasing. If the process of the change of mass consists of a finite number of function s(t) and besides if $s(t) \geqslant 0$, then the zero solution of equation (1.31) is stable while it is possible to obtain explicit limiting values of change of the angle and its derivative with respect to time.

Put $p(t) = \gamma\, q(t)$, where $\gamma = \text{const} > 0$. We write the final condition of formula (1.33) as

$$-\dot{q}\sqrt{\gamma} \geqslant q\gamma - s(t)| \qquad (1.35)$$

In order that (1.35) is satisfied we take:

$$q(t) = e^{-\frac{1}{\sqrt{\gamma}}\int_0^t [\gamma - s(t)]dt}$$

If there exists a positive number γ, such that the $\int_0^\infty |\gamma - s(t)|dt$ converges, then the zero solution of (1.31) is stable.

2. Motion of Sledge and Automobile of Variable Mass:

Consider the motion of sledge with variable mass along the horizontal plane, not taking into account the oscillation of the body (frame) [68]. The results obtained are applicable to an automobile as a particular case. The problem of tilting of automobile of constant mass has been examined by N.E. Zhukovskii[21], L.A. Chudakov[106], P.S. Linetkin and other authors.

1. Let the system (x,y) be fixed and set out in the horizontal plane of sledge motion. Denote the coordinates of inertial center of sledge C by x and y. We will neglect the displacement of point C about the sledge.

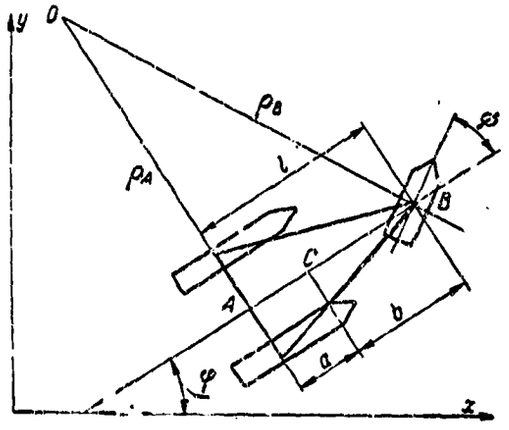

Рис. 15

Fig. 15

The position of the system is fixed by form coordinates x, y, φ and ψ, where φ is the angle between AB and then axis x, ψ is the angle of inclination of front shoes to the axis AB (see figure 15).

Coordinates of point A and B are respectively taken as x_1, y_1 and x_2, y_2. There are two non-holonomous constraints on the motion of sledge

$$F_1 = \dot{y}_1 \cos\varphi - \dot{x}_1 \sin\varphi = 0, \qquad (2.1)$$

$$F_2 = \dot{y}_2 \cos(\varphi+\psi) - \dot{x}_2 \sin(\varphi+\psi) = 0. \qquad (2.2)$$

Constraint (2.1) makes the velocity of point A in direction perpendicular to rear shoes equal to zero, constraint (2.2) makes the velocity of point B in a direction perpendicular to front

Kinetic energy of the sledge is written in the form

$$T = \frac{1}{2}\left[m(\dot{x}^2 + y^2) + I\varphi^2 + I_2\psi^2 + 2I_2\varphi\psi\right], \qquad (2.3)$$

Where m - mass of sledge,

I - moment of inertia of sledge about the vertical axis through the point C;

I_2 - moment of inertia of front

about the vertical axis through the point B.

m and I are known functions of time determined by the law of combustion. I_2 is constant.

Sledge moves under the action of force released by the change of mass with air resistance and friction of skiss . We will assume that the interval motion of particles in sledges and the process of their flow and the possible sucking in of air particles are symmetrical about the axis AB. Internal motion of particles brings into play carioliso forces proportional to $\dot{\varphi}$ and perpendicular to the axis AB. Therefore, the generalized reaction forces may be written in the form

$$\begin{aligned} \psi_x &= \psi_1 \cos\varphi - \psi_2 \dot{\varphi} \sin\varphi, \\ \psi_y &= \psi_1 \sin\varphi + \psi_2 \dot{\varphi} \cos\varphi, \\ \psi_\varphi &= -\psi_3 \dot{\rho}, \quad \psi_\rho = 0, \end{aligned} \tag{2.4}$$

where ψ_1, ψ_2 and ψ_3 are known functions of time. When the process of change of mass takes place on the surface (Meshcrky's case) functions ψ_2 and ψ_3 are equal to zero and ψ_1 is the force given by Meshersky's formula.

Generalized forces of resistance are

$$\begin{aligned} Q_x &= -Q_1 \cos\varphi + Q_2 \sin\varphi, \\ Q_y &= -Q_1 \sin\varphi - Q_2 \cos\varphi, \\ Q_\varphi &= -\frac{f_2 mgab}{l} \sin\psi - Q_3\dot{\varphi}, \\ Q_\psi &= -Q_4(\dot{\varphi}+\dot{\psi}) + L_\psi, \end{aligned} \tag{2.5}$$

where Q_1 and Q_2 are projections of principal vector of resistance on AB in the opposite direction and along a direction perpendicular to AB; f_2 o is resistance coefficient of the snow coating when the

front slides longitudinally; Q_3 and Q_4 are (constant) coefficients of air resistance when the systems linked with rear and front skis are rotated. L_ψ - moment on the axis of rudder.

We will suppose that air resistance during translational motion is proportional to the square of velocity with coefficient k. For the sake of convenience, we assume that the center of pressure coincides with the inertial center of sledge. Since the distances of the points A and C from the instantaneous center of velocities are $r_A = l \operatorname{ctg}\varphi$ and $r_C = \sqrt{a^2 + l^2 \operatorname{ctg}^2\varphi}$, we get

$$Q_1 = \frac{mg}{l}(af_2 \cos\varphi + bf_1) + kS\dot{\varphi}^2 (a^2 + l^2 \operatorname{ctg}^2\varphi), \quad (2.6)$$

$$Q_3 = \frac{f_2 mga}{l} \sin\varphi + \frac{kSa}{l}\dot{\varphi}^2 (a^2 + l^2 \operatorname{ctg}^2\varphi) \operatorname{tg}\psi. \quad (2.7)$$

Here S denotes the area of projection of sledge on a direction perpendicular to AB. f_1 - denotes the resistance coefficient during longitudinal sliding of real (skis).

Since

$$x_1 = x - a\cos\varphi, \quad x_2 = x + b\cos\varphi,$$

$$y_1 = y - a\sin\varphi, \quad y_2 = y + b\sin\varphi,$$

therefore, constraint equations (2.1) and (2.2) assume the form

$$F_1 = \dot{y}\cos\varphi - \dot{x}\sin\varphi - a\dot{\varphi} = 0,$$
$$F_2 = \dot{y}\cos(\varphi + \psi) - \dot{x}\sin(\varphi + \psi) + b\dot{\varphi}\cos\psi = 0. \quad (2.8)$$

2. Equations of motion and determination of reaction. We will find an expression for reaction and form the equations of motion by the method undetermined multipliers explained in 2, Chapter IV. We have

$$p'_x = m\dot{x},\quad p'_y = m\dot{y},\quad p'_\varphi = I\dot{\varphi} + I_2\dot{\psi},\quad p'_\psi = I_2(\dot{\varphi}+\dot{\psi}),$$

$$F'_1 = \frac{p'_y}{m}\cos\varphi - \frac{p'_x}{m}\sin\varphi - a\,\frac{p'_\varphi - p'_\psi}{I - I_2} = 0,$$

$$F'_2 = \frac{p'_y}{m}\cos(\varphi+\psi) - \frac{p'_x}{m}\sin(\varphi+\psi) + b\,\frac{p'_\varphi - p'_\psi}{I - I_2}\cos\psi = 0.$$

Put $I_1 = I - I_2$ and $\varkappa = \sqrt{\dfrac{I_1 + ma^2}{m}}$, then the matrix $\|A'_{kl}\|$ is written as

$$\begin{Vmatrix} \dfrac{\varkappa^2}{I_1} & \dfrac{\varkappa^2 - al}{I_1}\cos \\ \dfrac{\varkappa^2 - al}{I_1}\cos\varphi & \dfrac{\varkappa^2 - a^2}{I_1} + \dfrac{b^2}{I_1}\cos^2\psi \end{Vmatrix}$$

Therefore, the inverse matrix of coefficients $\|B'_{kl}\|$ is

$$\begin{Vmatrix} \dfrac{I_1(\varkappa^2 - a^2 + b^2\cos^2\psi)}{(\varkappa^2 - a^2)(\varkappa^2\sin^2\psi + l^2\cos^2\varphi)} & \dfrac{I_1(al - \varkappa^2)\cos}{(\varkappa^2 - a^2)(\varkappa^2\sin^2\varphi + l^2\cos^2\varphi)} \\ \dfrac{I_1(al - \varkappa^2)\cos}{(\varkappa^2 - a^2)(\varkappa^2\sin^2\varphi + l^2\cos^2\varphi)} & \dfrac{I_1\,\varkappa^2}{(\varkappa^2 - a^2)(\varkappa^2\sin^2\varphi + l^2\cos^2\varphi)} \end{Vmatrix}$$

Solve equations (2.8) for $\dot{x}$ and $\dot{y}$

$$\begin{aligned} \dot{x} &= (\,l\,\mathrm{ctg}\psi\cos\varphi - a\sin\varphi)\dot{\varphi}, \\ y &= (\,l\,\mathrm{ctg}\psi\sin\varphi + a\cos\varphi)\dot{\varphi}, \end{aligned} \qquad (2.9)$$

Then the special derivatives occuring in undetermined multipliers are obtained in the form

$$\frac{DF'_1}{Dt} = \frac{1}{m}(\quad_2 - Q_2) - \frac{a}{I_1}(\quad + Q\quad) - l\varphi^2 \operatorname{ctg}\quad,$$

$$\frac{DF'_2}{Dt} = \frac{1}{m}(\quad_1 - Q_1)\sin\quad + \frac{1}{m}(\quad_2 - Q_2)\cos\quad +$$

$$+ \frac{b}{I_1}(Q\quad + \quad)\cos\quad - \frac{I}{\sin\quad}(a\sin^2\quad + l\cos^2\quad)^2 + l\quad.$$

Undetermined multipliers are given by

$$\lambda_1 = m\left[b\varphi^2 + \frac{(al - \quad^2)l\varphi\quad}{\quad^2\sin^2\quad + l^2\cos^2\quad}\right]\operatorname{ctg}\quad + \frac{(al - \quad^2)(\quad_1 - Q_1)\sin\quad\cos\quad}{\quad^2\sin^2\quad + l^2\cos^2\quad} -$$

$$- \frac{[(\quad^2 - a^2)\sin^2\quad + bl\cos^2\quad](\quad_2 - Q_2) + (a\sin^2\quad + l\cos^2\quad)(\quad + Q\quad)}{\quad^2\sin^2\quad + l^2\cos^2\quad}, \qquad (2.10)$$

$$\lambda_2 = \frac{m}{\sin\quad}\left(a\varphi^2 + \frac{\quad^2 l\quad}{\quad^2\sin^2\quad + l^2\cos^2\quad}\right) +$$

$$+ \frac{\quad^2(\quad_1 - Q_1)\sin\quad - al(\quad_2 - Q_2)\cos\quad - l(\quad + Q\quad)\cos\quad}{\quad^2\sin^2\quad + l^2\cos^2\quad} \qquad (2.11)$$

In accordance with (2.3) - (2.5), equations of motion of sledge of variable mass are written as

$$m\ddot{x} = (\quad_1 - Q_1)\cos\quad - (\quad_2 - Q)\sin\quad + A_x,$$

$$m\ddot{y} = (\quad_1 - Q_1)\sin\quad + (\quad_2 - Q_2)\cos\quad + A_y,$$

$$I\ddot{\varphi} + I_2\ddot{\psi} = -\frac{f_1 mgab}{l}\sin\quad - (\quad_3 + Q_3)\quad + A_\varphi, \qquad (2.12)$$

$$I_2(\ddot{\varphi} + \ddot{\psi}) = -Q_4(\quad + \quad) + L_\psi,$$

According to analytical approach to constraints (2.8) for generalized reaction forces we get the formulae

$$A_x = -\lambda_1 \sin\varphi - \lambda_2 \sin(\varphi+\psi),$$

$$A_y = \lambda_1 \cos\varphi + \lambda_2 \cos(\varphi+\psi), \qquad (2.13)$$

$$A = -\lambda_1 a + \lambda_2 b \cos .$$

It follows that λ_1 and λ_2 in absolute values represent the lateral reactions of the points A and B.

3. Lateral stability of sledge and automobile of variable mass. In order to avoid the dangerous drift of skis i.e. slipping of rear skis in the direction perpendicular to AB, we must have

$$|\lambda_1| \leq \sigma_1 \frac{mgb}{l}, \qquad (2.14)$$

where σ_1 is the resistance coefficient of lateral shift of skis. To avoid slipping in direction perpendicular to front skis, the following condition must be satisified

$$|\lambda_2| \leq \sigma_2 \frac{mga}{l}. \qquad (2.15)$$

σ_2 is resistance coefficient of lateral shift of front skis.

According to (2.10) and the values

$$\sin\psi = \frac{l}{\sqrt{\rho_A^2 + l^2}}, \quad \cos\psi = \frac{\rho_A}{\sqrt{\rho_A^2 + l^2}}, \quad v_A = \rho_A \dot{\varphi},$$

condition (2.14) assumes the form

$$v_A^2 + v_1 v_A < \sigma_1 g \rho_A -$$

$$- \frac{\rho_A}{mb(\rho_A^2 + \varkappa^2)} \left[(ab - \varkappa^2) \rho_A (\Psi_1 - Q_1) - (b\rho_A^2 + \varkappa^2 l - a^2 l) \varkappa \right.$$

$$\left. \varkappa \left(\frac{v_A}{\rho_A} \Psi_2 - Q_2 \right) + (\rho_A^2 + al)(\Psi_\varphi + Q_\varphi) \right] . \qquad (2.16)$$

In this

$$v_1 = \frac{al - \varkappa^2}{bl} \frac{\rho_A^2 + l^2}{\rho_A^2 + \quad ^2} \rho_A \dot{\psi} .$$

Suppose that l^2 and x^2 are small as compared to ρ_A^2, then we get

$$v_1 = \frac{al - x^2}{bl} \rho_A \psi .$$

In accordance with (2.4) - (2.7) we will have

$$\left[mb + (a^2 - \varkappa^2) k\delta \right] v_A^2 + (mbv_1 - b\Psi_2 + \Psi_3 + Q_3) v_A \leqslant$$

$$\leqslant \sigma_1 g \quad _A - (al - \varkappa^2) \left[\Psi_1 - \frac{mg}{l} (af_2 \cos\psi - bf_1) \right] . \qquad (2.17)$$

Formulae (2.16) and (2.17) can be regarded as modification and refinement (great accuracy) in the conditions of lateral stability and automobile and sledge of variable mass. Similar details can be worked out for the condition (2.15).

It may be noted that the values of lateral reactions λ_1 and λ_2, on which depends the stability of sledge on turning are seen to depend on pull and the angular velocity of turning of the sudder. This accounts for the factor

$$\frac{(a_1 - x^2)\operatorname{ctg}\psi}{x^2\sin^2\psi + l^2\cos^2\psi}\left[ml\varphi\dot{\psi} + (\Psi_1 - Q_1)\sin^2\psi\right] \quad (2.18)$$

on the right side of (2.10)

On the right side of (2.11) we may take out factor of the type

$$\frac{x^2}{(x^2\sin^2\psi + l^2\cos^2\psi)\sin\psi}\left[ml\varphi\dot{\psi} + (\Psi_1 - Q_1)\sin^2\psi\right]. \quad (2.19)$$

The signs of (2.18) and (2.19) are the same as those of γ_1 and γ_2.

When the sledge or automobile turns three actions take place 1) first transient curve $\varphi > 0$; 2) motion with fixed rudder $\varphi = 0$ (in para 4 we will show that in this case sledge acquires circular motion in a circle of constant radius) 3) second transient curve $\varphi < 0$. From (2.18) and (2.19) it can be seen that on first transient curve reaction is greater than the value when $\varphi = 0$ whereas in second transient curve contrary to the first one it is less. Therefore, we can formulate the following law with reference to the sledges of variable mass.

It is called law of E.A. Chydakov. To maintain stability on turning at high speed reduce the pull on the motion of sledge along first transient curve and on the contrary increase it on motion of sledge along the second transient curve. The value of least A may be determined from the condition of lateral stability when the motion is circular.

According to last two equations of the system (2.12) we get

$$\dot{\varphi} = (\varkappa^2 \sin^2\psi + l^2\cos^2\psi)^{-1}\Big\{l^2\dot{\varphi}\dot{\psi}\operatorname{ctg}\psi + \frac{1}{m}\Big[l(\Psi_1 - Q_1)\cos\psi +$$

$$+ a(\Psi_2\dot{\varphi} - Q_2)\sin\psi + (\Psi_3\dot{\varphi} + Q_3\dot{\varphi})\sin\psi\sin\psi +$$

$$+ \frac{Q_4}{I_1}(\dot{\varphi} + \dot{\psi}) - \frac{L\dot{\psi}}{I_1}. \qquad (2.20)$$

According to (2.4) - (2.7), the terms in square bracket may be written in the form

$$l\Psi_1\cos\psi - mg(af_2 + bf_1\cos\psi) +$$

$$+ (a\Psi_2 - \Psi_3 - Q_3)\dot{\varphi}\sin\psi - \frac{k\delta}{l}\dot{\varphi}^2(a^2\operatorname{tg}\psi + l^2\operatorname{ctg}\psi)^2\cos\psi. \qquad (2.21)$$

Formula (2.20) shows that the pull Ψ_1 decrease the angle of rudder ψ : when $\dot{\psi}$ is positive it introduces an additional positive factor in the expression for $\dot{\varphi}$ and a negative $\dot{\psi}$ introduces a negative factor. From (2.20) - (2.21) we find the resistance to the sliding of skis and air resistance to the translational motion of sledge act in the opposite directions. They decrease $\dot{\varphi}$ when ψ is positive and increase with negative $\dot{\psi}$.

Since $\Psi_2\dot{\varphi}$ is principal vector of carioliso forces and $\Psi_3\dot{\varphi}$ is moment of these forces about the center of gravity of the sledge, then we have

$$a\Psi_2 - \Psi_3 \geqslant 0. \qquad (2.22)$$

Equality in (2.22) holds for surface combustion when $\Psi_2 = 0$, $\Psi_3 = 0$, and in the case when carioliso forces have a resultant at a point A. (2.20) and (2.21) show that carioliso forces increase the rotation in angle while the air snow resistance to the rotation of sledge

decrease the rotation in angle .

4. Nature of motion in special cases we will examine some special cases of motion of sledge and automobile with variable mass.

a) Suppose during the entire motion $\Psi = 0$. Last equation of system (2.12) gives

$$L\dot{\omega} = I_2\dot{\varphi} + Q_4\dot{\varphi}. \tag{2.23}$$

Since $l\dot{\varphi}\operatorname{ctg}\omega$ represents the velocity v_A and is consequently a boundry value, therefore, by equations (2.20) and (2.23) we get $\dot{\varphi} = \text{const}$. However, this constant may be zero as is indicated by the constraint equation (2.28), and thus we have $\varphi = \varphi_0 = \text{constant}$.

From equation (2.9) we find similarly that

$$y - y_0 = (x - x_0)\operatorname{tg}\varphi_0.$$

From this if follows that in the present case rectilinear motion of sledge takes place. Let $\varphi_0 = 0$, for the chosen system of coordinates, then the first equation of the system (2.12) according to formulae (2.6) and (2.13) is written as

$$m\ddot{x} = \Psi_1 - \frac{mg}{l}(af_2 + bf_1) - k\delta\dot{x}^2. \tag{2.24}$$

When the process of internal motion of particles in sledges sets in, the relative derivative of principal vector of relative momentum is equal to zero. If the mass changes without sucking in air particles, the force Ψ_1 is equal to the per second decrease of the momentum of particles.

$$\Psi_1 = -\frac{dm}{dt} u_r. \qquad (2.25)$$

Here u_r is relative velocity of flight of particles. It is seem from the equation (2,23) that rectilinear motion of sledge is equivalent to vertical ascending motion of point of variable mass if the acceleration of gravity is equal to $gl^{-1}(af_2 + bf_1)$ and the resistance coefficient is k.

Let u_r = constant, and the law of combustion be $f = mm_o^{-1}$ or linear $f = e^{-\beta t}$, or $f = 1 - \alpha t$. In the first case, put

$$c^2 = \frac{kS}{m_o}\left[\frac{g}{l}(af_2 + bf_1) - u_r\beta\right], \quad \nu^2 = 1,$$

in second case, let

$$c^2 = \frac{kS\varepsilon}{m_o l}(af_2 + bf_1), \quad \nu^2 = 4\frac{kSu_r}{m_o\alpha},$$

We may then obtain an expression (5) for the velocity of sledge in the form

$$\dot{x} = \frac{m_o}{kS} f^{\frac{1}{2}} \left\{\left(\frac{1}{2}\frac{\eta}{\eta} - \frac{\nu}{\xi\eta\frac{1}{2}}\right) f^{\frac{1}{2}} + C\frac{I_{\nu-1}(\xi) + DN_{\nu-1}(\xi)}{I_\nu(\xi) + DN_\nu(\xi)}\right\} \qquad (2.26)$$

In this,

$$\xi = C\int f^{-\frac{1}{2}}\,dt, \qquad \eta = f^{-\frac{1}{2}}\int f^{-\frac{1}{2}}\,dt,$$

I_ν and N_ν are Bessel function of first and second order D an arbitrary constant. (B) Suppose the motion takes place with constant angle of rudder $\psi = \psi_o$ = constant, Radius of curvature of trajectories of the points A and B will be

$$\rho_A = l \operatorname{ctg} \psi_0 = \text{const}, \quad \rho_B = \frac{l}{\sin \psi_0} = \text{const}.$$

Therefore, the sledge must move in a circle of constant radius. Integrating the constraints equations (2.9) we get

$$\begin{aligned} x &= C_1 + l \operatorname{ctg} \psi_0 \sin \varphi + a \cos \varphi, \\ y &= C_2 - l \operatorname{ctg} \psi_0 \cos \varphi + a \sin \varphi. \end{aligned} \tag{2.27}$$

Constants C_1 and C_2 are equal to

$$C_1 = x_0 - l \operatorname{ctg} \psi_0 \sin \varphi_0 - a \cos \varphi_0,$$

$$C_2 = y_0 + l \operatorname{ctg} \psi_0 \cos \varphi_0 - a \sin \varphi_0.$$

In accordance with (2.27) we have

$$(x - C_1)^2 + (y - C_2)^2 = \rho_C^2.$$

Consequently C_1 and C_2 are regarded as the coordinates of the center of the circle.

Last equation of the system (2.12) will assume the form (2.23). For motion at angle φ we get according to (2.20), (2.21) and (2.23)

$$\ddot{\varphi} = \frac{\left[l \Psi_1 \cos \psi_0 - mg (af_2 + bf_1 \cos \psi_0) + (a \Psi_2 - \Psi_3 - Q_3) \dot{\varphi} \sin \psi_0 \sin \psi_0 \right] \frac{2}{m (x^2 \sin^2 \psi_0 + l^2 \cos^2 \psi_0)}}{}$$

$$- \frac{\frac{k\Delta}{l} \dot{\varphi}^2 (a^2 \operatorname{tg} \psi_0 + l^2 \operatorname{ctg} \psi_0)^2 \cos \psi_0 \sin \psi_0}{m (x^2 \sin^2 \psi_0 + l^2 \cos^2 \psi_0)} \tag{2.28}$$

In order that the circular motion of the sledge is uniform it is necessary and sufficient that the right side of the expression (2.28) tends to zero. Since Ψ_2 Ψ_3 and Q_3 are considerably smaller than Ψ_1

and any air resistance, therefore with an angular velocity $\dot{\varphi}$ not very large the condition is equivalent to the requirement

$$\Psi_1 = \frac{mg}{l \cos \psi_0} (af_2 + bf_1 \cos \psi_0) + \frac{kS}{l^2} v_C^2 (l^2 + a^2 tg^2 \psi_0). \tag{2.29}$$

When the angle ψ_0 is small, we get

$$\Psi_1 = \frac{mg}{l} (af_2 + bf_1) + kSv_A^2.$$

Strictly speaking, when condition (2.29) is satisfied we do get uniform rotation because the angular velocity is equal to

$$\dot{\varphi} = \dot{\varphi}_0 e^{\int_0^t \frac{(a\Psi_2 - \Psi_3 - Q_3) \sin^2 \psi_0}{m (x^2 \sin^2 \psi_0 + l^2 \cos^2 \psi_0)} dt}$$

Angular $\dot{\varphi}$ velocity will be constant only

When $$\Psi_2 = \frac{1}{a} (\Psi_3 + Q_3). \tag{2.30}$$

Suppose the construction of sledge is such that the relation between I_1 and m remains constant during the entire motion, then x = constant. If the formula (2.25) holds good for the pull (or draught) then under condition (2.30) or in the terms $a\Psi_2 - \Psi_3 - Q_3$, equation (2.24) can be written in the form

$$m\ddot{\varphi} = -\frac{dm}{dt} u_r' - mg' - k' S \dot{\varphi}^2. \tag{2.31}$$

Here

$$u_r' = \frac{l u_r \sin \psi_o \cos \psi_o}{x^2 \sin^2 \psi_o + l^2 \cos^2 \psi_o},$$

$$g' = \frac{g (af_2 + bf_1 \cos \psi_o) \sin \psi_o}{x^2 \sin^2 \psi_o + l^2 \cos^2 \psi_o},$$

$$k' = \frac{k (a^2 \operatorname{tg} \psi_o + l^2 \operatorname{ctg} \psi_o)^2 \sin \psi_o \cos \psi_o}{l (x^2 \sin^2 \psi_o + l^2 \cos^2 \psi_o)}$$

Equation (2.31) can be discussed in the same manner as equation (2.24).

Suppose the rudder is not fixed at an angle ψ_o but its axis supports moment L_ψ, determined by (2.23). Last equation of the system (2.12) assumes the form

$$I_2 \dot{\psi} = - Q_4 \dot{\psi}.$$

Denote by ψ_B and $\dot{\psi}_B$ the perturbations of the angle and the angular velocity of the rudder at a certain moment t_B for the $t \geqslant t_B$ we get

$$\psi - \psi_o = \psi_B + \frac{I_2 \dot{\psi}_B}{Q_4} \left[1 - e^{- \frac{Q_4}{I_2} (t - t_B)} \right]$$

It is, therefore, concluded that motion of sledge in a circle of constant radius has stability of its own kind.

V) Suppose the angle of rudder can be regarded as a given function of time $\psi = \psi (t)$. Taking into consideration last equation of the system (2.21), equation (2.20) can be written as

$$\varphi + f_1(t) \dot{\varphi} + f_2(t) \dot{\varphi}^2 = f_3(t).$$

Here

$$f_1(t) = -\frac{I_1\left[(a\dot{\psi}_2 - \dot{\psi}_3 - Q_3)\sin^2\psi + ml^2\psi\operatorname{ctg}\psi\right]}{mI\,(x^2\sin^2\psi + l^2\cos^2\psi)},$$

$$f_2(t) = \frac{kSI_1\,(a^2\operatorname{tg}\psi + l^2\operatorname{ctg}\psi)^2\sin\psi\cos\psi}{mll\,(x^2\sin^2\varphi + l^2\cos^2\psi)},$$

$$f_3(t) = \frac{I_1\left[l\dot{\psi}_1\cos\psi - mg\,(af_2 + bf_1\cos\psi)\sin\psi\right]}{mI\,(x^2\sin^2\psi + l^2\cos^2\psi)} - \frac{I_2\dot{\psi}}{I}$$

If we put $k = 0$, then the problem of motion of sledge is solved in quadrature. Actually we have

$$\varphi_{k=0} = e^{-\int_0^t f_1(\tau)\,d\tau}\left[\varphi_0 + \int_0^t f_3(\tau)\,e^{\int_0^{\tau} f_1(t_1)\,dt_1}\,d\tau\right]. \tag{2.32}$$

Formula (2.32) gives the angular velocity of rotation of a stream lined sledge. In the general case $|\dot{\varphi}| \leq |\dot{\varphi}_{k=0}|$. Integration (2.29), we find the expression for the coordinates of center of gravity of sledge of variable mass.

g) We will examine the motion of sledge of variable mass with free rudder when $L\psi = 0$. Last equation of the system (2.12) takes the form

$$I_2(\dot{\varphi} + \dot{\psi}) = -Q_4(\varphi + \psi). \tag{2.33}$$

we thus get

$$\varphi + \psi = \varphi_0 + \psi_0 + \frac{I_2(\dot{\varphi}_0 + \dot{\psi}_0)}{Q_4}\left[1 - e^{-\frac{Q_4}{I_2}(t-t_0)}\right].$$

When $t \longrightarrow \infty$, we have

$$\varphi + \psi = \varphi_0 + \psi_0 + \frac{I_2 (\dot{\varphi}_0 + \dot{\psi}_0)}{Q_4} . \qquad (2.34)$$

Suppose the angle of front skis attains the value (2.34) then $\dot{\varphi} + \dot{\psi} = 0$. Since $S_A = l\varphi \operatorname{ctg}\psi$, where S_A is the path which could be adopted by the point A, we find

$$\sin \psi = e^{-\frac{S_A}{l}} \sin \psi_0 .$$

Therefore, $\psi = 0$ as $S_A \to \infty$.

If we make the rudder free, then the sledges of variable mass move in rectilinear motion the direction of which is determined by formula (2.34).

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